

RANKIN-SELBERG L -FUNCTIONS IN CYCLOTOMIC TOWERS, III

JEANINE VAN ORDER

ABSTRACT. Let π be a cuspidal automorphic representation of GL_2 over a totally real number field F . Let K be a quadratic extension of F . Fix a prime ideal of F , and consider the set X of all finite-order Hecke characters of K unramified outside of this prime. We consider averages of central values of the Rankin-Selberg L -functions $L(s, \pi \times \pi(\mathcal{W}))$, where \mathcal{W} ranges over characters in X , and $\pi(\mathcal{W})$ denotes the representation of GL_2 associated to a $\mathcal{W} \in X$. In particular, we estimate the averages over such characters $\mathcal{W} \in X$ which arise from Dirichlet characters after composition with the norm homomorphism from K to the rational number field, and determine that these do not vanish when the conductor of \mathcal{W} is sufficiently large. When the representation π is assumed to be a holomorphic discrete series of weight greater than one, and also the quadratic extension K/F is assumed to be totally imaginary, we can derive stronger results extending those of Rohrlich via either the algebraicity theorem of Shimura or else the existence of a related p -adic interpolation series. Thus our results also apply to show the nontriviality of various constructions of p -adic interpolation series that appear in the literature, which had previously only been conjectured.

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1. INTRODUCTION

The aim of this article, in continuation with the prequels [30] and [31], is to establish generic nonvanishing properties for certain families of central values of Rankin-Selberg L -functions for $\mathrm{GL}_2 \times \mathrm{GL}_2$ over a totally real number fields. Such results have various applications to Iwasawa theory, in particular to showing the nontriviality of related p -adic L -function constructions, and then from this the nontriviality of related Euler system constructions once reciprocity laws have been established (whence bounds for Mordell-Weil ranks of elliptic curves can be obtained unconditionally, for instance). The nontriviality of these p -adic L -functions in turn can be used to derive stronger results about the nonvanishing of the complex central values in families. For instance, we can derive a general analogue of Rohrlich's theorem [27] for cuspidal automorphic representations of GL_2 over a totally real number field. Such results are also of interest on their own from the point of view of automorphic L -functions, and suggest that there might be eventually be a theory of algebraic Satake parameters to account for this generic nonvanishing behaviour.

Let F be a totally real number field of degree $d = [F : \mathbf{Q}]$, integer ring \mathcal{O}_F , and adele ring \mathbf{A}_F . Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p and residue degree $\delta = \delta_{\mathfrak{p}} = [F_{\mathfrak{p}} : \mathbf{Q}_p]$. Fix a quadratic extension of number fields K/F of relative discriminant $\mathfrak{D} = \mathfrak{D}_{K/F}$ and associated idele class character $\eta = \eta_{K/F}$. We consider the set $\mathfrak{X}(\mathfrak{p})$ of finite-order Hecke characters of K which are unramified outside of \mathfrak{p} . In particular, we shall consider the subset $X(\mathfrak{p}) \subset \mathfrak{X}(\mathfrak{p})$ of characters of the form $\mathcal{W} = \rho\psi$, where ρ is a ring class character of some \mathfrak{p} -power conductor, and $\psi = \xi \circ \mathbf{N}_{K/F} = \chi \circ \mathbf{N}$ is a cyclotomic character of some \mathfrak{p} -power conductor.¹ To be more precise, each character $\mathcal{W} = \rho\psi \in X(\mathfrak{p})$ can be described as follows. The ring class character ρ can be viewed as a character of the class group of the \mathcal{O}_F -order $\mathcal{O}_{\mathfrak{p}^\alpha} = \mathcal{O}_F + \mathfrak{p}^\alpha \mathcal{O}_K$ of conductor \mathfrak{p}^α in K for some integer $\alpha \geq 0$. Thus,

$$\rho : \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha}) = \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^\alpha}^\times \longrightarrow \mathbf{C}$$

is a unitary idele class character of K .² Here, we use standard conventions in writing \mathbf{A}_K to denote the adele ring of K , \mathbf{A}_K^\times the idele group of K , and $K_\infty = K \otimes \mathbf{C}$ the archimedean component, along with the shorthand “hat” notation for the profinite completion. The conductor $c(\rho)$ of this character ρ is given by the ideal $\mathfrak{p}^\alpha \subset \mathcal{O}_F$. The cyclotomic character $\psi = \xi \circ \mathbf{N}_{K/F} = \chi \circ \mathbf{N}$ is obtained after composition with the norm homomorphism $\mathbf{N} = \mathbf{N}_{K/\mathbf{Q}}$ from K to \mathbf{Q} with a Dirichlet character χ of p -power conductor (conductor p^β say). Here, for notational convenience, we write $\xi = \chi \circ \mathbf{N}_{F/\mathbf{Q}}$ to denote the idele class character of F obtained by composing with the norm homomorphism $\mathbf{N}_{F/\mathbf{Q}}$ from F to \mathbf{Q} . Hence this character ξ is related to the idele class character ψ of K by the formula $\psi = \xi \circ \mathbf{N}_{K/F}$, where $\mathbf{N}_{K/F}$ denotes the norm homomorphism from K to F , and has conductor $c(\xi) = \mathfrak{p}^\beta \subset \mathcal{O}_F$. To each character $\mathcal{W} \in X(\mathfrak{p})$, we can associate an automorphic representation $\pi(\mathcal{W}) = \otimes_v \pi(\mathcal{W})_v$ of $\mathrm{GL}_2(\mathbf{A}_F)$. Thus it makes sense to consider the $\mathrm{GL}_2 \times \mathrm{GL}_2$

¹Leopoldt's conjecture, which is known for F abelian, would imply that $X(\mathfrak{p}) = \mathfrak{X}(\mathfrak{p})$.

²Composing with the reciprocity map of class field theory, such a character ρ factors through the Galois group of the ring class field $K[\mathfrak{p}^\alpha]$ of conductor \mathfrak{p}^α over K , which is of generalized dihedral type over F , and moreover linearly disjoint over K to the cyclotomic \mathbf{Z}_p -extension of K . For this reason, such a character ρ is often said to be *dihedral* or *anticyclotomic*.

Rankin-Selberg L function associated to π and $\pi(\mathcal{W})$ for any character $\mathcal{W} \in X(\mathfrak{p})$,

$$L(s, \pi \times \mathcal{W}) = L(s, \pi \times \pi(\mathcal{W})) = \prod_{v \nmid \infty} L(s, \pi_v \times \pi(\mathcal{W})_v).$$

This degree-four L -function has an analytic continuation by the theory of Jacquet [11] and Jacquet-Langlands [12], which we recall in some modest detail below. Its completed L -function $\Lambda(s, \pi \times \pi(\mathcal{W}))$ satisfies a functional equation of the form

$$(1) \quad \Lambda(s, \pi \times \mathcal{W}) = \epsilon(s, \pi \times \mathcal{W}) \Lambda(1-s, \tilde{\pi} \times \mathcal{W}^{-1}),$$

where $\tilde{\pi}$ denotes the contragredient of π , and the ϵ -factor $\epsilon(s, \pi \times \mathcal{W})$ evaluated at the central point $s = 1/2$ defines a complex number of modulus one $\epsilon(1/2, \pi \times \mathcal{W}) \in \mathbf{S}^1$ known as the *root number* of $L(s, \pi \times \mathcal{W})$. This root number admits a decomposition

$$\epsilon(1/2, \pi \times \mathcal{W}) = \epsilon(1/2, \pi \times \pi(\mathcal{W})) = \prod_v \epsilon(1/2, \pi_v \times \pi(\mathcal{W})_v)$$

into local root numbers $\epsilon(1/2, \pi_v \times \pi(\mathcal{W})_v)$, and can be given explicitly in this setting by a precise formula involving a fourth power of a Gauss sum associated to the cyclotomic part $\psi = \chi \circ \mathbf{N}$ of \mathcal{W} . We refer to the discussion below for details on the exact form (i.e. (4)). It will suffice for the purposes of this introduction to know that it is not in general real-valued (in fact never for $\beta \geq 2$), whence the functional equation (1) does not in general force any vanishing of the central values $L(1/2, \pi \times \mathcal{W})$.

The main purpose of this work is to determine to the best extent possible how seldom the values $L(1/2, \pi \times \mathcal{W})$ vanish as $\mathcal{W} = \rho\psi$ varies over characters of the set $X(\mathfrak{p})$, and in particular over the cyclotomic characters ψ . Let us first consider this problem in its most classical form, as one about averages of central values of Rankin-Selberg L -functions. Given integers $\alpha, \beta \geq 0$, let us write $X_{\alpha, \beta}$ to denote the subset of characters of $X(\mathfrak{p})$ of the form $\mathcal{W} = \rho\psi = \rho\xi \circ \mathbf{N}_{K/F}$, where the ring class part ρ has conductor $c(\rho)$ dividing \mathfrak{p}^α , and the cyclotomic part ξ has conductor $c(\xi)$ dividing \mathfrak{p}^β (whence $X(\mathfrak{p}) = \bigcup_{\alpha, \beta \geq 0} X_{\alpha, \beta}$). Similarly, let us write X_β to denote the set of cyclotomic characters $\psi = \xi \circ \mathbf{N}_{K/F}$ with conductor dividing \mathfrak{p}^β . We shall consider the averages defined by first moments, i.e.

$$H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = H_{\pi, \mathfrak{D}}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = |X_{\alpha, \beta}|^{-1} \sum_{\mathcal{W} \in X_{\alpha, \beta}} L(1/2, \pi \times \mathcal{W})$$

and (for $\rho \in X(\mathfrak{p})$ a fixed ring class character)

$$H(\mathfrak{p}^\beta) = H_{\pi, \mathfrak{D}, \rho}(\mathfrak{p}^\beta) = |X_\beta|^{-1} \sum_{\psi \in X_\beta} L(1/2, \pi \times \rho\psi).$$

We first show the following result for these averages. Roughly, we start with the approximate functional equation to describe the values $L(1/2, \pi \times \mathcal{W})$, then use that the $X_{\alpha, \beta}$ and X_β define full orthogonal sets of characters to obtain exact formulae for $H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ and $H(\mathfrak{p}^\beta)$ (Theorem 4.2). Using these exact formulae, we can reduce the problem of estimation to some relatively elementary computations with hyper-Kloosterman sums of prime power modulus.³

³This result also generalizes the one shown in [30, Corollary 1.3] for the classical setting with modular elliptic curves defined over the rational number field, i.e. to the setting of arbitrary cuspidal automorphic representations of GL_2 over totally real number fields (cf. also [32]).

Theorem 1.1 (Theorem 4.6, Corollary 4.7). *Assume for simplicity that the relative discriminant $\mathfrak{D} = \mathfrak{D}_{K/F}$ is prime to \mathfrak{p} . (i) If the exponent $\beta \geq 0$ is sufficiently large, then for each choice of exponent $\alpha \geq 0$, the average $H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = H_{\pi, \mathfrak{D}}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ does not vanish. (ii) If the exponent $\beta \geq 0$ is sufficiently large, then for each choice of ring class character $\rho \in X(\mathfrak{p})$, of K the average $H(\mathfrak{p}^\beta) = H_{\pi, \mathfrak{D}, \rho}(\mathfrak{p}^\beta)$ does not vanish.*

It is possible to obtain from this result an analogous proposition for primitive averages after using the Möbius inversion formula. Thus let us for integers $\alpha, \beta \geq 0$ write $P_{\alpha, \beta}$ to denote the subset of characters $\mathcal{W} = \rho \xi \circ \mathbf{N}_{K/F} \in X(\mathfrak{p})$ with ρ a primitive ring class character of conductor \mathfrak{p}^α , and $\psi = \xi \circ \mathbf{N}_{K/F} = \chi \circ \mathbf{N}$ whose underlying Dirichlet character χ is primitive of conductor p^β . Similarly, let P_β denote the subset of such characters $\psi = \xi \circ \mathbf{N}_{K/F} = \chi \circ \mathbf{N}$. We then define

$$\mathcal{P}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = \mathcal{P}_{\pi, \mathfrak{D}}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = |P_{\alpha, \beta}|^{-1} \sum_{\mathcal{W} \in P_{\alpha, \beta}} L(1/2, \pi \times \mathcal{W})$$

and (for $\rho \in X(\mathfrak{p})$ a fixed ring class character)

$$\mathcal{P}(\mathfrak{p}^\beta) = \mathcal{P}_{\pi, \mathfrak{D}, \rho}(\mathfrak{p}^\beta) = |P_\beta|^{-1} \sum_{\psi \in P_\beta} L(1/2, \pi \times \rho\psi).$$

Corollary 1.2 (Corollary 5.2). *(i) If the exponent $\beta \geq 0$ is sufficiently large, then for each choice of exponent $\alpha \geq 0$, the average $H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = H_{\pi, \mathfrak{D}}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ does not vanish. (ii) If the exponent $\beta \geq 0$ is sufficiently large, then for each choice of ring class character $\rho \in X(\mathfrak{p})$, the average $H(\mathfrak{p}^\beta) = H_{\pi, \mathfrak{D}, \rho}(\mathfrak{p}^\beta)$ does not vanish.*

Note that we have not imposed many conditions on the representation $\pi = \otimes_v \pi_v$ so far, nor on the choice of quadratic extension K/F . The second purpose of this work is to derive stronger nonvanishing results in the setting where (i) π is a holomorphic discrete series of weight $(k_j)_{j=1}^d$ with each $k_j \geq 2$, and (ii) the quadratic extension K/F is assumed to be totally imaginary. When (i) and (ii) hold, we can invoke the following important theorem due to Shimura [28]. In short, for each finite-order Hecke character K of \mathcal{W} , it is shown in [28] that the value $\mathcal{L}(1/2, \pi \times \mathcal{W}) = L(1/2, \pi \times \mathcal{W}) \langle \pi, \pi \rangle^{-1}$ is algebraic, where $\langle \pi, \pi \rangle$ denotes the Petersson norm of π . More precisely, writing $\mathbf{Q}(\pi)$ to denote the finite extension of \mathbf{Q} obtained by adjoining the eigenvalues of π , and $\mathbf{Q}(\pi, \mathcal{W})$ the finite extension of $\mathbf{Q}(\pi)$ obtained by adjoining the values of \mathcal{W} , the values $\mathcal{L}(1/2, \pi \times \mathcal{W})$ lie in $\mathbf{Q}(\pi, \mathcal{W})$. Moreover, these algebraic values are Galois conjugate in the sense that there is an action of $\sigma \in \text{Aut}(\mathbf{C})$ on them via the rule

$$\sigma(\mathcal{L}(1/2, \pi \times \mathcal{W})) = \mathcal{L}(1/2, \pi^\sigma \times \mathcal{W}^\sigma).$$

Here, π^σ denotes the representation of $\text{GL}_2(\mathbf{A}_F)$ obtained from π by applying σ to its eigenvalues, and \mathcal{W}^σ the character defined on nonzero ideals $\mathfrak{a} \subset \mathcal{O}_K$ by the rule $\mathfrak{a} \mapsto \mathcal{W}(\mathfrak{a})^\sigma$. Restricting to embeddings σ of $\mathbf{Q}(\pi, \mathcal{W})$ into \mathbf{C} which fix $\mathbf{Q}(\pi)$, we obtain a Galois conjugate family of values $\mathcal{L}(1/2, \pi \times \mathcal{W})$, i.e. where the action fixes π but varies over Galois conjugate characters \mathcal{W} (i.e. \mathcal{W} of the same order). When $\mathbf{Q}(\pi)$ is linearly disjoint over \mathbf{Q} to the cyclotomic extension $\mathbf{Q}(\mathcal{W})$ obtained by adjoining the values of \mathcal{W} , we then define

$$(2) \quad G_{[\mathcal{W}]}(\pi) = [\mathbf{Q}(\pi, \mathcal{W}) : \mathbf{Q}(\pi)]^{-1} \sum_{\sigma: \mathbf{Q}(\pi, \mathcal{W}) \rightarrow \mathbf{C}} L(1/2, \pi \times \mathcal{W}^\sigma).$$

Here, the sum runs over complex embeddings $\sigma : \mathbf{Q}(\pi, \mathcal{W}) \rightarrow \mathbf{C}$ which fix $\mathbf{Q}(\pi)$. These averages are of interest to estimate from the point of view of nonvanishing, since the Galois conjugacy implies that either all or none of the summands vanish as \mathcal{W}^σ varies over automorphisms of $\mathbf{Q}(\pi, \mathcal{W})$ fixing $\mathbf{Q}(\pi)$, cf. e.g. the arguments of Rohrlich [27]. In this direction, we can deduce the following result.

Corollary 1.3 (Corollary 5.3). *Assume that K/F is totally imaginary, and that π is a holomorphic discrete series of weight $(k_j)_{j=1}^d$ with each $k_j \geq 2$. Fix a ring class character $\rho \in X(\mathfrak{p})$ of K . Assume that the Hecke field $\mathbf{Q}(\pi)$ is linearly disjoint over \mathbf{Q} to the cyclotomic field $\mathbf{Q}(\rho)$, as well as to the cyclotomic tower $\mathbf{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbf{Q}(\zeta_{p^n})$. If the exponent $\beta \geq 0$ is sufficiently large, then for each cyclotomic character $\xi = \chi \circ \mathbf{N} \in P_\beta$, the Galois average $G_{[\rho\chi \circ \mathbf{N}]}(\pi)$ does not vanish. Thus for all but finitely many cyclotomic characters $\psi = \xi \circ \mathbf{N}_{K/F} = \chi \circ \mathbf{N}$ whose underlying Dirichlet character χ has p -power conductor, the central value $L(1/2, \pi \times \rho\psi) = L(1/2, \pi \times \rho\xi \circ \mathbf{N}_{K/F}) = L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ does not vanish.*

We can also use the existence of a closely related $(\delta+1)$ -variable p -adic L -function (Theorem 5.4 below) to obtain the following results in the same direction. Let us now fix an embedding $\iota_p : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$.

Corollary 1.4 (Theorem 5.6). *Assume that K/F is totally imaginary, that π is a holomorphic discrete series of weight $(k_j)_{j=1}^d$ with each $k_j \geq 2$, and for simplicity that π is \mathfrak{p} -ordinary (i.e. that the image under ι_p of its eigenvalue for the Hecke operator at \mathfrak{p} is a p -adic unit). Let $\rho \in X(\mathfrak{p})$ be any ring class character of K of \mathfrak{p} -power conductor. Then, for all but finitely many Dirichlet characters χ of p -power conductor, the central value $L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ does not vanish.*

We also derive the following result for the twisted L -functions of π in this setting. To be more precise, let us write $L(s, \pi)$ to denote the L -function of π , and $L(s, \pi, \xi)$ to denote the L -function of π twisted by an idele class character ξ of F .

Corollary 1.5 (Corollary 5.7). *For all but finitely many Dirichlet characters χ of p -power conductor, the central values $L(1/2, \pi, \chi \circ \mathbf{N}_{F/\mathbf{Q}})$ and $L(1/2, \pi, \eta\chi \circ \mathbf{N}_{F/\mathbf{Q}})$ do not vanish.*

Thus, we derive a general analogue of the theorem of Rohrlich [27] in this setting. Note however that we do not attempt in this work to extend the other theorem of Rohrlich [26] for the self-dual setting (i.e. averaging over Galois conjugate ring class characters ρ in the classical setting of CM elliptic curves over imaginary quadratic fields). This latter problem turns out to be subtle from the perspective of analytic averaging, and we address this in [32]. We can however obtain the following result in this direction by using a variation of the main basechange argument given in [31]. This argument amounts to checking that the p -adic L -functions we consider are compatible in a formal way with the basechange of π in the sense of [14] and [1] to certain cyclic and solvable extensions of the field F , whence we can use Artin formalism for the associated L -values to deduce the following.

Theorem 1.6 (Theorem 5.9). *Assume that K/F is totally imaginary, that π is a holomorphic discrete series of weight $(k_j)_{j=1}^d$ with each $k_j \geq 2$, and for simplicity that π is \mathfrak{p} -ordinary. There exists in this setting an effectively computable integer $\beta_0 \geq 0$ such that for all cyclotomic characters $\psi = \chi \circ \mathbf{N} \in X(\mathfrak{p})$ of conductor greater than \mathfrak{p}^{β_0} , the central value $L(1/2, \pi \times \rho\psi)$ does not vanish for any ring class character ρ in the set $X(\mathfrak{p})$.*

Anyhow, we give two methods of establishing the nontriviality of Galois averages here. One comes from the algebraicity theorem of Shimura [28] and the other from the existence of a related p -adic L -function (see e.g. Theorem 5.4 below). This latter method allows us to give a softer proof, but is essentially equivalent to invoking algebraicity theorems of [28] (which by no coincidence are prerequisite for these p -adic L -functions constructions). Thus, the algebraicity properties of the values $L(1/2, \pi \times \mathcal{W})$ (or rather their weighted counterparts $\mathcal{L}(1/2, \pi \times \mathcal{W})$) appear to be key for deriving such results. This viewpoint suggests in particular that such nonvanishing results should not be restricted to certain families Rankin-Selberg L -values for $\mathrm{GL}_2 \times \mathrm{GL}_2$, but rather that they should appear as a general phenomenon for automorphic L -functions with algebraic Satake parameters.

Although we do not discuss the implications at length in this work, the results outlined above show the nontriviality of various p -adic L -functions constructions, most notably those of Dabrowski [6], Dimitrov [7], Hida [10], Mok [20], and Spiess [24] (which had not been known previously, even granted the theorem of Rohrlich [25]). It would be of interest to explore the implications for Hida families in this direction, and we hope to take the issue up in subsequent work(s).

1.0.1. Outline. We first give a brief account of the theory of the Rankin-Selberg L -functions we consider in §2. We then derive a formula for our central values using the approximate functional equation method in §3. Using this formula, along with explicit descriptions of the coefficients and orthogonality of characters, we then derive exact formulae for the averages we consider (Theorem 4.2). We then evaluate the various contributions explicitly to obtain our analytic estimates (i.e. Theorem 4.6 and Corollary 4.7). The final section §5 describes how to refine these estimates to obtain information about (i) averages over primitive characters (Corollary 5.2), (ii) averages over Galois conjugate (cyclotomic) characters via Shimura algebraicity (Corollary 5.3), and (iii) averages over Galois conjugate (cyclotomic) characters via the existence of a related p -adic L -function (Theorem 5.6 and Corollary 5.7).

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2. RANKIN-SELBERG L -FUNCTIONS

Let us first review the theory of Jacquet [11] and Jacquet-Langlands [12] briefly for the Rankin-Selberg L -functions of $\mathrm{GL}_2(\mathbf{A}_F)$ we consider, using the conventions and notations given in [13, §5]. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ with central character $\omega = \otimes_v \omega_v$. Fix a quadratic extension K of F with associated idele class character $\eta = \otimes_v \eta_v$ of F . Let \mathcal{W} be a Hecke character of K , and $\pi(\mathcal{W}) = \otimes_v \pi(\mathcal{W})_v$ its associated automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. We shall consider the Rankin-Selberg L -function of π and $\pi(\mathcal{W})$, written here as

$$L(s, \pi \times \mathcal{W}) = L(s, \pi \times \pi(\mathcal{W})) = \prod_{v \nmid \infty} L(s, \pi_v \times \pi(\mathcal{W})_v).$$

This Euler product over nonarchimedean places v converges absolutely for $\Re(s) > 1$. Here, for primes v not dividing the conductor $c(\pi \times \mathcal{W})$ of $L(s, \pi \times \mathcal{W})$, the local

factor $L(s, \pi_v \times \mathcal{W}_v)$ is defined by the formula

$$L(s, \pi_v \times \mathcal{W}_v) = \det(I - A_v \otimes B_v \mathbf{N}v^{-s})^{-1},$$

where A_v denotes the Satake parameter of π_v and B_v that of $\pi(\mathcal{W})_v$. For primes v where one of π or $\pi(\mathcal{W})$ is ramified, the local factor $L(s, \pi_v \times \pi(\mathcal{W})_v)$ takes the form of $P_v(\mathbf{N}v^{-s})^{-1}$ for $P_v(x)$ a polynomial of degree at most four such that $P_v(0) = 1$. Let us write $L(s, \pi_\infty \times \mathcal{W}_\infty)$ as shorthand to denote the archimedean component, which takes the form

$$L(s, \pi_\infty \times \mathcal{W}_\infty) = \prod_{j=1}^2 \prod_{k=1}^2 \Gamma_F(s - \mu_\infty(j) - \nu_\infty(k)), \quad \Gamma_F(s) = \left(\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right)^d.$$

We refer to the excellent monograph [19, §1.1.2] for more details. Let us note as well that this L -function $L(s, \pi \times \mathcal{W})$ is equivalent to the basechange L -function $L(s, \pi_K, \mathcal{W})$, i.e. where π_K denotes the basechange of π to $\mathrm{GL}_2(\mathbf{A}_K)$ and $L(s, \pi_K, \mathcal{W})$ the L -function of π_K twisted by the Hecke character \mathcal{W} of K .

The completed L -function

$$\Lambda(s, \pi \times \mathcal{W}) = L(s, \pi \times \mathcal{W}) L(s, \pi_\infty \times \mathcal{W}_\infty)$$

is entire unless $\pi(\mathcal{W}) \approx \tilde{\pi} \otimes |\cdot|^t$ for some $t \in \mathbf{R}$, in which case it is holomorphic except for simple poles at $s = 0$ and 1 . It satisfies a functional equation of the form

$$\Lambda(s, \pi \times \mathcal{W}) = \epsilon(s, \pi \times \mathcal{W}) L(1-s, \tilde{\pi} \times \overline{\mathcal{W}}).$$

The ϵ -factor $\epsilon(s, \pi \times \mathcal{W})$ evaluated at the central point $s = 1/2$ defines a complex number of modulus one known as the *root number* $\epsilon(1/2, \pi \times \mathcal{W})$, which admits an Euler product decomposition

$$(3) \quad \epsilon(1/2, \pi \times \mathcal{W}) = \prod_v \epsilon(1/2, \pi_v \times \mathcal{W}_v) = \prod_v \epsilon(1/2, \pi_v \times \pi(\mathcal{W})_v).$$

2.1. Root number formula. Let us now give a formula for the root number (3) in our setting (cf. e.g. [16], [17, 2.1], and [5, §1]). Let us assume for simplicity that we take a Hecke character \mathcal{W} in the set $X(\mathfrak{p})$ defined above. Hence, \mathcal{W} has some \mathfrak{p} -power conductor $c(\mathcal{W}) \subset \mathcal{O}_F$, and its restriction to \mathbf{A}_F^\times is given by $\eta\xi^2$ for ξ some idele class character of F . Let us also assume for simplicity that (i) the prime-to- \mathfrak{p} part $c(\pi)'$ of the level $c(\pi)$ of π is prime to the relative discriminant \mathfrak{D} , and (ii) the central character ω of π is unramified at \mathfrak{p} . We then have the formula

$$(4) \quad \epsilon(1/2, \pi \times \mathcal{W}) = \omega(c(\mathcal{W})) \eta \xi^2 (c(\pi)') \epsilon(1/2, \pi) \tau(\eta \xi^2)^4 \mathbf{N}c(\eta \xi^2)^{-2}.$$

Here, $\epsilon(1/2, \pi)$ denotes the root number of the L -function $L(s, \pi)$ associated to π , which is defined in the analogous way with respect to the functional equation $\Lambda(s, \pi) = \epsilon(s, \pi) \Lambda(1-s, \pi)$ of the completed L -function $\Lambda(s, \pi)$ of $L(s, \pi)$. Also, $c(\eta \xi^2) \subset \mathcal{O}_F$ denotes the conductor of the idele class character $\eta \xi^2$ of F , and $\tau(\eta \xi^2)$ its Gauss sum, which can be defined as follows (see [8, Kap. VIII] or [18, Ch. VII]). Write $\mathfrak{d} = \mathfrak{d}_F$ to denote the different of F , and fix a class $y \in c(\eta \xi^2) \mathfrak{d}^{-1}$. Then

$$\tau(\eta \xi^2) = \sum_{\substack{x \bmod c(\eta \xi^2) \\ (x, c(\eta \xi^2))=1}} \eta \xi^2(x) \exp(\mathrm{Tr}(xy) 2\pi i),$$

where the sum runs over any fixed set of representatives x of $(\mathcal{O}_F/c(\eta\xi^2)\mathcal{O}_F)^\times$. An equivalent definition can be given in terms of idele numbers (see e.g. the discussion in [18, Ch. VII, (7.4)]), though we do not require such a description here.

2.2. Dirichlet series expansions. We shall use the following equivalent Dirichlet series expansions for $L(s, \pi \times \mathcal{W})$. Let us always take $\Re(s) > 1$ for this discussion. Let us also commit an abuse of notation in writing \mathbf{N} to denote both the absolute norm and also the norm homomorphism $\mathbf{N} = \mathbf{N}_{K/F}$ from K to F , taking it on faith that the context will make the distinction clear. The first expansion comes essentially from the basechange equivalence $L(s, \pi \times \mathcal{W}) = L(s, \pi_K, \mathcal{W})$. That is,

$$L(s, \pi \times \mathcal{W}) = L(s, \pi_K, \mathcal{W}) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \lambda_\pi(\mathbf{N}\mathfrak{a}) \mathcal{W}(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-s},$$

where the sum runs over nonzero integral ideals $\mathfrak{a} \subset \mathcal{O}_K$ whose image under the norm homomorphism $\mathbf{N}\mathfrak{a} = \mathbf{N}_{K/F}\mathfrak{a} \subset \mathcal{O}_F$ is prime to the conductor $c(\mathcal{W}) \subset \mathcal{O}_F$. On the other hand, decomposing the Hecke character \mathcal{W} into its dihedral and cyclotomic parts as $\mathcal{W} = \rho\psi = \rho\xi \circ \mathbf{N}_{K/F} = \rho\chi \circ \mathbf{N}$, we also have the expansion

(5)

$$L(s, \pi \times \mathcal{W}) = \sum_{\mathfrak{m} \subset \mathcal{O}_F} \omega\eta\xi^2(\mathfrak{m}) \mathbf{N}\mathfrak{m}^{-2s} \sum_{\mathfrak{n} \subset \mathcal{O}_F} \lambda_\pi(\mathfrak{n}) \xi(\mathfrak{n}) \left(\sum_{A \in \text{Pic}(\mathcal{O})} r_A(\mathfrak{n}) \rho(A) \right) \mathbf{N}\mathfrak{n}^{-s}.$$

Here, $\text{Pic}(\mathcal{O}) = \text{Pic}(\mathcal{O}_{c(\rho)})$ is the class group of the \mathcal{O}_F -order $\mathcal{O}_{c(\rho)} := \mathcal{O}_F + c(\rho)\mathcal{O}_K$ of conductor $c(\rho) \subset \mathcal{O}_F$ in K , and $r_A(\mathfrak{n})$ the number of ideals in the class A whose image under $\mathbf{N}_{K/F}$ equals \mathfrak{n} .

3. APPROXIMATE FUNCTIONAL EQUATION

Let us for $\Re(s) > 1$ write the Dirichlet series expansion of $L(s, \pi \times \mathcal{W})$ as

$$L(s, \pi \times \mathcal{W}) = \sum_{\mathfrak{n} \subset \mathcal{O}_F} a_{\pi \times \mathcal{W}}(\mathfrak{n}) \mathbf{N}\mathfrak{n}^{-s}.$$

The purpose of this section is to obtain a convenient expression for the value at the central point $s = 1/2$ via the approximate functional equation method. Let us to this end fix a meromorphic test function $G(s)$ on \mathbf{C} such that

- $G(s)$ is holomorphic except at $s = 0$, where $G(s) = 1/s + O(1)$ as $s \rightarrow 0$,
- $G(s)$ is polynomial growth on vertical lines,
- $G(-s) = G(s)$ for all $s \neq 0$.

We then define from $G(s)$ a cutoff function $V(y) \in \mathcal{C}_\infty(\mathbf{R}_{>0})$ by the relation

$$V(y) = \int_{\Re(s)=2} \widehat{V}(s) y^{-s} \frac{ds}{2\pi i},$$

where for $s \in \mathbf{C} - \{0\}$ we put

$$(6) \quad \widehat{V}(s) = L_\infty(1/2, \pi \times \mathcal{W})^{-1} L_\infty(s + 1/2, \pi \times \mathcal{W}) G(s).$$

The idea now is to apply a standard contour argument to the integral

$$\int_{\Re(s)=2} \Lambda(s + 1/2, \pi \times \mathcal{W}) G(s) \frac{ds}{2\pi i}.$$

Using the properties of the test function $G(s)$ with the functional equation satisfied by the completed L -function $\Lambda(s, \pi \times \mathcal{W})$, it is not hard to derive the formula

$$\begin{aligned} L(1/2, \pi \times \mathcal{W}) &= \sum_{\mathfrak{n} \subset \mathcal{O}_F} a_{\pi \times \mathcal{W}}(\mathfrak{n}) \mathbf{N}\mathfrak{n}^{-\frac{1}{2}} V\left(\frac{\mathbf{N}\mathfrak{n}}{\sqrt{c(\pi \times \mathcal{W})}}\right) \\ &\quad + \epsilon(1/2, \pi \times \mathcal{W}) \sum_{\mathfrak{n} \subset \mathcal{O}_F} a_{\tilde{\pi} \times \mathcal{W}^{-1}}(\mathfrak{n}) \mathbf{N}\mathfrak{n}^{\frac{1}{2}} - V\left(\frac{\mathbf{N}\mathfrak{n}}{\sqrt{c(\pi \times \mathcal{W})}}\right). \end{aligned}$$

Since the derivation is very standard (cf. [13, §5.2] or [30, Proposition 2.1]), we omit the details here, and take this formula for granted throughout.⁴ We shall also take for granted the fact that V decays rapidly in the following precise sense.

Lemma 3.1. *Let $j \geq 0$ be any integer, and $C \geq 0$ any constant. The j -th derivative $V^{(j)}$ of the cutoff function V defined above satisfies the following growth properties, i.e. with the convention that $V^{(0)} = V$.*

$$V^{(j)}(y) = \begin{cases} (\log y)^{(j)} + O_j(y^{\frac{1}{2}} - j) & \text{if } 0 \leq y \leq 1 \\ O_{C,j}(y^{-C}) & \text{if } y \geq 1. \end{cases}$$

Proof. The result is established by direct computation, cf. e.g. [29, Lemma 7.1]. \square

4. AVERAGE VALUES

Let us now fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p . We shall consider a Hecke character \mathcal{W} of K of the type described above, admitting the decomposition(s) $\mathcal{W} = \rho\psi = \rho\xi \circ \mathbf{N}_{K/F} = \rho\chi \circ \mathbf{N}$. Again, ρ is a ring class character of K of some conductor $c(\rho) \subset \mathcal{O}_F$, which we shall choose to be $c(\rho) = \mathfrak{p}^\alpha$ for $\alpha \geq 0$ an integer. Also, $\xi = \chi \circ \mathbf{N}_{F/\mathbf{Q}}$ is what we have called a “cyclotomic” Hecke character of the field F , whose conductor $c(\xi) \subset \mathcal{O}_F$ we shall chose to be $c(\xi) = \mathfrak{p}^\beta$ for $\beta \geq 1$ an integer. Thus writing $\delta = [F_{\mathfrak{p}} : \mathbf{Q}_p]$, the absolute norms of these conductors are given by $\mathbf{N}c(\rho) = \mathbf{N}\mathfrak{p}^\alpha = p^{\delta\alpha} = p^a$ and $\mathbf{N}c(\xi) = \mathbf{N}\mathfrak{p}^\beta = p^{\delta\beta} = p^b$. Writing $q = \mathfrak{p}^\beta$, and $c = \mathfrak{p}^\alpha$ as shorthand, we consider the averages of central values

(7)

$$H(c, q) = H_{\pi, \mathfrak{D}}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = \frac{1}{\varphi(p^\beta)h(\mathcal{O}_{\mathfrak{p}^\alpha})} \sum_{\xi = \chi \circ \mathbf{N} \bmod \mathfrak{p}^\beta} \sum_{\rho \in \text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})^\vee} L(1/2, \pi \times \rho\chi \circ \mathbf{N})$$

and (for ρ fixed)

$$(8) \quad H(q) = H_{\pi, \mathfrak{D}, \rho}(q) = \frac{1}{\varphi(p^\beta)} \sum_{\xi = \chi \circ \mathbf{N} \bmod \mathfrak{p}^\beta} L(1/2, \pi \times \rho\chi \circ \mathbf{N}).$$

Here, $h(\mathcal{O}_c) = |\text{Pic}(\mathcal{O}_c)|$ denotes the class number of the \mathcal{O}_F -order $\mathcal{O}_c = \mathcal{O}_F + c\mathcal{O}_K$ of conductor $c \subset \mathcal{O}_F$ of K , which is given by Dedekind’s formula

$$(9) \quad h(\mathcal{O}_c) = \frac{h(\mathcal{O}_K)[\mathcal{O}_K : \mathcal{O}_c]}{[\mathcal{O}_K^\times : \mathcal{O}_c^\times]} \prod_{c'|c} \left(1 - \frac{\eta(c')}{\mathbf{N}c'}\right).$$

⁴Essentially, the theorems of Cauchy and Stirling justify moving the line of integration to $\Re(s) = -2$, crossing a residue (the L -value) at $s = 0$. The formula is then easy to verify by using the functional equation for $\Lambda(-s + 1/2, \pi \times \mathcal{W})$ and the properties of the chosen function $G(s)$.

4.1. Exact formulae. We seek to establish a formulae for each of the averages (7) and (8). To achieve this, we use the approximate functional equation formula given above for the values $L(1/2, \pi \times \rho\chi \circ \mathbf{N})$. We then use partial summation and orthogonality of characters in each case to obtain the following result, whose proof is nearly identical to the one given for the classical setting in [30]. To state the obtained formula, let us first establish some simplifying notation for the square root of the conductor $c(\pi \times \mathcal{W}) = c(\pi \times \rho\chi \circ \mathbf{N})$ of $L(s, \pi \times \rho\chi \circ \mathbf{N})$. That is, we can assume without loss of generality that $c(\pi \times \mathcal{W})^{\frac{1}{2}} = Mp^{2a}p^{2b} = Mp^{2(a+b)}$ for M an integer depending only upon the absolute norms $\mathbf{N}c(\pi)$ and $\mathbf{N}\mathfrak{D} = \mathbf{N}\mathfrak{D}_{K/F}$ (cf. e.g. the discussion in [25]). In particular, this integer M does not vary with the choice of character $\mathcal{W} \in X(\mathfrak{p})$. Thus, our approximate functional equation formula for the values $L(1/2, \pi \times \mathcal{W}) = L(1/2, \pi \times \rho\psi)$ takes the form

$$\begin{aligned} L(1/2, \pi \times \mathcal{W}) &= \sum_{\mathbf{n} \subset \mathcal{O}_F} a_{\pi \times \mathcal{W}}(\mathbf{n}) \mathbf{N}\mathbf{n}^{-\frac{1}{2}} V\left(\frac{\mathbf{N}\mathbf{n}}{Mp^{2(a+b)}}\right) \\ &\quad + \epsilon(1/2, \pi \times \mathcal{W}) \sum_{\mathbf{n} \subset \mathcal{O}_F} a_{\tilde{\pi} \times \mathcal{W}^{-1}}(\mathbf{n}) \mathbf{N}\mathbf{n}^{\frac{1}{2}} V\left(\frac{\mathbf{N}\mathbf{n}}{Mp^{2(a+b)}}\right). \end{aligned}$$

To evaluate the averages (8) and (7) using the formula above, we must first apply either one or two rounds of partial summation respectively to deal with the fact that $c(\pi \times \mathcal{W})^{\frac{1}{2}} = Mp^{2(a+b)}$ does vary with \mathcal{W} . Thus, let $X_{\alpha, \beta}$ denote the set of all Hecke characters $\mathcal{W} = \rho\xi \circ \mathbf{N}_{K/F} \in X(\mathfrak{p})$ of K whose ring class factor ρ has conductor $c(\rho)$ dividing \mathfrak{p}^α , and whose cyclotomic factor $\xi \circ \mathbf{N}_{K/F}$ has conductor $c(\xi)$ dividing \mathfrak{p}^β . Here, $\alpha \geq 0$ and $\beta \geq 0$ are taken to be arbitrary positive integers. Let us also write $P_{\alpha, \beta}$ to denote the set of all Hecke characters $\mathcal{W} = \rho\xi \circ \mathbf{N}_{K/F} = \rho\chi \circ \mathbf{N}$ whose ring class part ρ is primitive of conductor $c(\rho) = \mathfrak{p}^\alpha$, and whose cyclotomic part $\xi \circ \mathbf{N}_{K/F}$ is primitive of conductor \mathfrak{p}^β . We have the elementary decomposition

$$(10) \quad X_{\alpha, \beta} = \bigcup_{\substack{0 \leq x \leq \alpha \\ 0 \leq y \leq \beta}} P_{x, y},$$

which induces decompositions of the sums over coefficients appearing in (7),

$$(11) \quad \sum_{\rho\chi \circ \mathbf{N} \in X_{\alpha, \beta}} \sum_{\mathbf{n} \subset \mathcal{O}_F} a_{\pi \times \rho\chi \circ \mathbf{N}}(\mathbf{n}) = \sum_{\substack{0 \leq x \leq \alpha \\ 0 \leq y \leq \beta}} \sum_{\rho'\chi' \circ \mathbf{N} \in P_{x, y}} \sum_{\mathbf{n} \subset \mathcal{O}_F} a_{\pi \times \rho'\chi' \circ \mathbf{N}}(\mathbf{n}).$$

The single variable sums (8) can be treated in a completely analogous way, writing X_β to denote the set of Dirichlet characters $\chi \circ \mathbf{N} \bmod \mathfrak{p}^\beta$, and P_β the subset of primitive Dirichlet characters $\bmod \mathfrak{p}^\beta$ to obtain the decomposition

$$(12) \quad \sum_{\chi \circ \mathbf{N} \in X_\beta} \sum_{\mathbf{n} \subset \mathcal{O}_F} a_{\pi \times \rho\chi \circ \mathbf{N}}(\mathbf{n}) = \sum_{0 \leq y \leq \beta} \sum_{\chi' \circ \mathbf{N} \in P_y} \sum_{\mathbf{n} \subset \mathcal{O}_F} a_{\pi \times \rho\chi' \circ \mathbf{N}}(\mathbf{n}).$$

Now, to apply partial summation to (12) and (11), we first write the sums over primitive characters of the coefficients $a_{\pi \times \rho'\chi' \circ \mathbf{N}}(\mathbf{n})$ for a fixed nonzero ideal $\mathbf{n} \subset \mathcal{O}_F$ as

$$g_{\mathbf{n}, x, y} = \sum_{\rho'\chi' \circ \mathbf{N} \in P_{x, y}} a_{\pi \times \rho'\chi' \circ \mathbf{N}}(\mathbf{n})$$

and

$$g_{\mathbf{n},y} = \sum_{\chi' \circ \mathbf{N} \in P_y} a_{\pi \times \rho \chi' \circ \mathbf{N}}(\mathbf{n}).$$

We then write the respective sums over exponents as

$$G_{\mathbf{n}}(\alpha, \beta) = \sum_{\substack{0 \leq x \leq \alpha \\ 0 \leq y \leq \beta}} g_{\mathbf{n},x,y}$$

and

$$G_{\mathbf{n}}(\beta) = \sum_{0 \leq y \leq \beta} g_{\mathbf{n},y}.$$

Let us also write $g_{\mathbf{n},x,y}^*$, $g_{\mathbf{n},y}^*$, $G_{\mathbf{n}}^*(\alpha, \beta)$, and $G_{\mathbf{n}}^*(\beta)$ to denote the sums defined in the analogous way with respect to the contragredient coefficients $\epsilon(1/2, \pi \times \mathcal{W}) a_{\tilde{\pi} \times \overline{\mathcal{W}}}(\mathbf{n})$. Recall that we define the cutoff function V on the interval $y \in (0, \infty)$ by

$$V(y) = \int_{\Re(s)=2} \widehat{V}(s) y^{-s} \frac{ds}{2\pi i},$$

where $\widehat{V}(s)$ is defined in (6) above. Let us define from this a related functions \mathfrak{V}_j on for $j = 1, 2$ by the rule

$$\mathfrak{V}_j(y) = \int_{\Re(s)=2} (1 - \mathbf{Np}^{-2s}) \widehat{V}(s) \left(\frac{y}{\mathbf{Np}^{2j}} \right) \frac{ds}{2\pi i}.$$

Applying two rounds and one round of partial summation respectively to (12) and (11) gives the following basic formulae (cf. [30, Corollary 3.2]).

Lemma 4.1. *We have the following formulae for the averages (7) and (8) in terms of the naive Dirichlet series expansion $L(s, \pi \times \rho \chi \circ \mathbf{N}) = \sum_{\mathbf{n} \subset \mathcal{O}_F} a_{\pi \times \rho \chi \circ \mathbf{N}}(\mathbf{n}) \mathbf{Nn}^{-s}$.*

(i) *The average $H(c, q) = H_{\pi}(\mathbf{p}^{\alpha}, \mathbf{p}^{\beta})$ defined in (7) is given by the expression*

$$\begin{aligned} & \sum_{\mathbf{n} \subset \mathcal{O}_F} \left(G_{\mathbf{n}}(\alpha, \beta) V \left(\frac{\mathbf{Nn}}{Mp^{2(a+b)}} \right) + E_{\mathbf{n}}(\alpha, \beta) \right) \mathbf{Nn}^{-\frac{1}{2}} \\ & + \sum_{\mathbf{n} \subset \mathcal{O}_F} \left(G_{\mathbf{n}}^*(\alpha, \beta) V \left(\frac{\mathbf{Nn}}{Mp^{2(a+b)}} \right) + E_{\mathbf{n}}^*(\alpha, \beta) \right) \mathbf{Nn}^{-\frac{1}{2}}, \end{aligned}$$

where

$$\begin{aligned} E_{\mathbf{n}}(\alpha, \beta) &= \sum_{\substack{0 \leq x \leq \alpha-1 \\ 0 \leq y \leq \beta-1}} G_{\mathbf{n}}(x, y) \mathfrak{V}_2 \left(\frac{\mathbf{Nn}}{Mp^{2(x+y)}} \right) \\ &- \sum_{0 \leq x \leq \alpha-1} G_{\mathbf{n}}(x, \beta) \mathfrak{V}_1 \left(\frac{\mathbf{Nn}}{Mp^{2(x+b)}} \right) - \sum_{0 \leq y \leq \beta-1} G_{\mathbf{n}}(\alpha, y) \mathfrak{V}_1 \left(\frac{\mathbf{Nn}}{Mp^{2(a+y)}} \right) \end{aligned}$$

and

$$\begin{aligned}
E_n^*(\alpha, \beta) &= \sum_{\substack{0 \leq x \leq \alpha-1 \\ 0 \leq y \leq \beta-1}} G_n^*(x, y) \mathfrak{V}_2 \left(\frac{\mathbf{Nn}}{Mp^{2(x+y)}} \right) \\
&\quad - \sum_{0 \leq x \leq \alpha-1} G_n^*(x, \beta) \mathfrak{V}_1 \left(\frac{\mathbf{Nn}}{Mp^{2(x+b)}} \right) - \sum_{0 \leq y \leq \beta-1} G_n^*(\alpha, y) \mathfrak{V}_1 \left(\frac{\mathbf{Nn}}{Mp^{2(a+y)}} \right).
\end{aligned}$$

(ii) The average $H(q) = H_\pi(\mathfrak{p}^\beta)$ defined in (8) is given by the expression

$$\begin{aligned}
&\sum_{\mathbf{n} \subset \mathcal{O}_F} \left(G_n(\beta) V \left(\frac{\mathbf{Nn}}{Mp^{2(a+b)}} \right) - E_n(\beta) \right) \mathbf{Nn}^{-\frac{1}{2}} \\
&\quad + \sum_{\mathbf{n} \subset \mathcal{O}_F} \left(G_n^*(\beta) V \left(\frac{\mathbf{Nn}}{Mp^{2(a+b)}} \right) - E_n^*(\beta) \right) \mathbf{Nn}^{-\frac{1}{2}},
\end{aligned}$$

where

$$E_n(\beta) = \sum_{0 \leq y \leq \beta-1} G_n(\alpha, y) \mathfrak{V}_1 \left(\frac{\mathbf{Nn}}{Mp^{2(a+y)}} \right)$$

and

$$E_n^*(\alpha, \beta) = \sum_{0 \leq y \leq \beta-1} G_n^*(\alpha, y) \mathfrak{V}_1 \left(\frac{\mathbf{Nn}}{Mp^{2(a+y)}} \right).$$

Proof. The result of (i) follows from two applications of partial summation, and (ii) from one application. See the arguments of [30, Lemma 3.1, Corollary 3.2, and Proposition 5.1], each of which carries over in a direct way. \square

We can now evaluate the coefficients in these expressions via orthogonality to obtain the following exact formulae for the averages (12) and (11) (cf. [30, §§ 3, 5]). To do this in a succinct way, let us first establish the following intricate definitions.

Definition Keep all of the conventions and hypotheses above, in particular so that $\mathcal{W} = \rho\psi = \rho\xi \circ \mathbf{N}_{K/F} = \rho\chi \circ \mathbf{N} \in X(\mathfrak{p})$ is a Hecke character of K with ring class part ρ of conductor $c = c(\rho) = \mathfrak{p}^\alpha$ and cyclotomic part $\psi = \xi \circ \mathbf{N}_{K/F} = \chi \circ \mathbf{N}$ of conductor $q = \mathfrak{p}^\beta$. Let us for simplicity write N to denote the conductor $c(\pi) \subset \mathcal{O}_F$, as well as $Q = \mathfrak{D}q$ to denote the conductor $c(\eta\xi^2) \subset \mathcal{O}_F$ of the idele class Dirichlet character of F defined by the product $\eta\xi^2$. We also write $h_{\alpha,\beta}$ to denote the cardinality of the underlying set of characters, i.e. $h_{\alpha,\beta} = h(\mathcal{O}_{\mathfrak{p}^\alpha})\varphi(p^\beta)$, where $h(\mathcal{O}_{\mathfrak{p}^\alpha})$ denotes the cardinality of the class group $\text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})$, as well as $h_\beta = \varphi(p^\beta)$.

Let us for the double averages (7) consider the sums defined by

$$D(c, q) = \sum_{\mathbf{m} \subset \mathcal{O}_F} \frac{\omega\eta(\mathbf{m})}{\mathbf{Nm}} \sum_{\substack{\mathbf{n} \subset \mathcal{O}_F \\ \mathbf{m}^2 \mathbf{n} \equiv 1 \pmod{q}}} \frac{\lambda_\pi(\mathbf{n}) r_\alpha(\mathbf{n})}{\mathbf{Nn}^{\frac{1}{2}}} V \left(\frac{\mathbf{Nm}^2 \mathbf{Nn}}{Mp^{2(a+b)}} \right),$$

and

$$\begin{aligned} D^*(c, q) &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)}{\mathbf{N}Q^2} \\ &\times \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \subset \mathcal{O}_F} \frac{\lambda_\pi(\mathfrak{n})r_\alpha(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}}\right) \\ &\times \sum_{\substack{x_1, x_2, x_3, x_4 \bmod Q \\ \mathfrak{m}^2 \mathfrak{n} N^2 x_1^2 x_2^2 x_3^2 x_4^2 \equiv 1 \bmod q}} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4). \end{aligned}$$

Here, $r_\alpha(\mathfrak{n})$ is the number of nonzero ideals $\mathfrak{a} \subset \mathcal{O}_K$ of relative norm $\mathbf{N}_{K/F}(\mathfrak{a}) = \mathfrak{n}$ whose image in the class group $\text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})$ is trivial, and $\mathbf{e}(x) = \exp(\text{Tr}(xy)2\pi i)$ denotes the exponential function defined with respect to our fixed choice of class $y \in Q\mathfrak{d}^{-1} \subset \mathcal{O}_F$. Additionally, writing \mathfrak{V}_j to denote the function defined above for $j \in \{1, 2\}$, we shall consider the related sums

$$\mathfrak{D}_j(c, q) = \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{\mathfrak{n} \subset \mathcal{O}_F \\ \mathfrak{m}^2 \mathfrak{n} \equiv 1 \bmod q}} \frac{\lambda_\pi(\mathfrak{n})r_\alpha(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} \mathfrak{V}_j\left(\frac{\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}}\right)$$

and

$$\begin{aligned} \mathfrak{D}_j^*(c, q) &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)}{\mathbf{N}Q^2} \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \subset \mathcal{O}_F} \frac{\lambda_\pi(\mathfrak{n})r_\alpha(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} \mathfrak{V}_j\left(\frac{\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}}\right) \\ &\times \sum_{\substack{x_1, x_2, x_3, x_4 \bmod Q \\ \mathfrak{m}^2 \mathfrak{n} N^2 x_1^2 x_2^2 x_3^2 x_4^2 \equiv 1 \bmod q}} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4). \end{aligned}$$

Let us for the single averages (8) consider analogously the defined sums $D(q)$, $D^*(q)$, $\mathfrak{D}_j(q)$, and $\mathfrak{D}_j^*(q)$ by fixing the c -sum. That is, fixing the ring class character ρ of conductor $c = c(\rho) = \mathfrak{p}^\alpha$, let us for a nonzero ideal $\mathfrak{n} \subset \mathcal{O}_F$ write $r_\rho(\mathfrak{n})$ to denote the coefficient defined by

$$(13) \quad r_\rho(\mathfrak{n}) = \sum_{A \in \text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})} \rho(A) r_A(\mathfrak{n}),$$

where each $r_A(\mathfrak{n})$ is the number of ideals \mathfrak{a} in a fixed class $A \in \text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})$ of relative norm $\mathbf{N}_{K/F}(\mathfrak{a}) = \mathfrak{n}$. Hence, the sums $D(q)$, $D^*(q)$, $\mathfrak{D}_j(q)$, and $\mathfrak{D}_j^*(q)$ are defined in the same way, only that the coefficients $r_\alpha(\mathfrak{n})$ are replaced by these $r_\rho(\mathfrak{n})$.

Theorem 4.2. *Keep the setup above, so that $\mathcal{W} = \rho\psi = \rho\xi \circ \mathbf{N}_{K/F} = \rho\chi \circ \mathbf{N} \in X(\mathfrak{p})$ is a Hecke character of K with ring class part ρ of conductor $c = c(\rho) = \mathfrak{p}^\alpha$ and cyclotomic part $\psi = \xi \circ \mathbf{N}_{K/F} = \chi \circ \mathbf{N}$ of conductor $q = \mathfrak{p}^\beta$.*

(i) *The double average $H(c, q) = H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ of (7) is given by the formula*

$$H(c, q) = D(c, q) + D^*(c, q) + (\mathfrak{E}(c, q) + \mathfrak{E}^*(d, q)).$$

Here, $\mathfrak{E}(c, q)$ is defined by the difference of sums

$$h_{\mathfrak{p}^\alpha, \mathfrak{p}^\beta}^{-1} \left(\sum_{\substack{0 \leq x \leq \alpha-1 \\ 0 \leq y \leq \beta-1}} h_{\mathfrak{p}^x, \mathfrak{p}^y} \mathfrak{D}_2(\mathfrak{p}^x, \mathfrak{p}^y) - \sum_{0 \leq x \leq \alpha-1} h_{\mathfrak{p}^x, \mathfrak{p}^\beta} \mathfrak{D}_1(\mathfrak{p}^x, \mathfrak{p}^\beta) - \sum_{0 \leq y \leq \beta-1} h_{\mathfrak{p}^\alpha, \mathfrak{p}^y} \mathfrak{D}_1^*(\mathfrak{p}^\alpha, \mathfrak{p}^y) \right),$$

and $\mathfrak{E}^(c, q)$ by the difference*

$$h_{\mathfrak{p}^\alpha, \mathfrak{p}^\beta}^{-1} \left(\sum_{\substack{0 \leq x \leq \alpha-1 \\ 0 \leq y \leq \beta-1}} h_{\mathfrak{p}^x, \mathfrak{p}^y} \mathfrak{D}_2^*(\mathfrak{p}^x, \mathfrak{p}^y) - \sum_{0 \leq x \leq \alpha-1} h_{\mathfrak{p}^x, \mathfrak{p}^\beta} \mathfrak{D}_1^*(\mathfrak{p}^x, \mathfrak{p}^\beta) - \sum_{0 \leq y \leq \beta-1} h_{\mathfrak{p}^\alpha, \mathfrak{p}^y} \mathfrak{D}_1^*(\mathfrak{p}^\alpha, \mathfrak{p}^y) \right).$$

(ii) The single average $H(c) = H(\mathfrak{p}^\beta) = H_\rho(\mathfrak{p}^\beta)$ of (8), i.e. for ρ a fixed ring class character of conductor $c = c(\rho) = \mathfrak{p}^\alpha$, is given by the formula

$$H(q) = D(q) + D^*(q) - (\mathfrak{E}(q) + \mathfrak{E}^*(q)).$$

Here, $\mathfrak{E}(q)$ is defined by

$$h_{\mathfrak{p}^\beta}^{-1} \sum_{0 \leq y \leq \beta-1} h_{\mathfrak{p}^y} \mathfrak{D}_1(\mathfrak{p}^\alpha, \mathfrak{p}^y),$$

and $\mathfrak{E}^*(q)$ by

$$h_{\mathfrak{p}^\beta}^{-1} \sum_{0 \leq y \leq \beta-1} h_{\mathfrak{p}^y} \mathfrak{D}_1^*(\mathfrak{p}^\alpha, \mathfrak{p}^y).$$

Proof. We follow the style of argument of [30, Proposition 3.4]. Let us first consider the average $H(c, q) = H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$, which from the discussion above is given by the sum $S_1 + S_2 + S_3 + S_4$, with

$$\begin{aligned} S_1 &= \sum_{\mathfrak{n} \subset \mathcal{O}_F} G_{\mathfrak{n}}(\alpha, \beta) V \left(\frac{\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}} \right) \mathbf{N}\mathfrak{n}^{-\frac{1}{2}} \\ S_2 &= \sum_{\mathfrak{n} \subset \mathcal{O}_F} E_{\mathfrak{n}}(\alpha, \beta) \mathbf{N}\mathfrak{n}^{-\frac{1}{2}} \\ S_3 &= \sum_{\mathfrak{n} \subset \mathcal{O}_F} G_{\mathfrak{n}}^*(\alpha, \beta) V \left(\frac{\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}} \right) \mathbf{N}\mathfrak{n}^{-\frac{1}{2}} \\ S_4 &= \sum_{\mathfrak{n} \subset \mathcal{O}_F} E_{\mathfrak{n}}^*(\alpha, \beta) \mathbf{N}\mathfrak{n}^{-\frac{1}{2}}. \end{aligned}$$

Let us start with the contribution of S_1 , which in the Dirichlet series expansion $\sum_{\mathfrak{a} \subset \mathcal{O}_K} \lambda_\pi \circ \mathbf{N}_{K/F}(\mathfrak{a}) \rho \chi \circ \mathbf{N}(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-s}$ over nonzero ideals of \mathcal{O}_K is given by

$$\begin{aligned} & \sum_{\mathfrak{a} \subset \mathcal{O}_K} \sum_{\rho \in \text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})^\vee} \sum_{\chi \bmod p^\beta} \rho(\mathfrak{a}) \chi(\mathbf{N}\mathfrak{a}) \lambda_\pi(\mathbf{N}_{K/F}(\mathfrak{a})) V \left(\frac{\mathbf{N}\mathfrak{a}}{Mp^{2(a+b)}} \right) \mathbf{N}\mathfrak{a}^{-\frac{1}{2}} \\ &= \varphi(p^\beta) \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ \mathbf{N}\mathfrak{a} \equiv 1 \bmod p^\beta}} \sum_{\rho \in \text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})^\vee} \rho(\mathfrak{a}) \lambda_\pi(\mathbf{N}_{K/F}(\mathfrak{a})) V \left(\frac{\mathbf{N}\mathfrak{a}}{Mp^{2(a+b)}} \right) \mathbf{N}\mathfrak{a}^{-\frac{1}{2}} \\ &= \varphi(p^\beta) h(\mathcal{O}_{\mathfrak{p}^\alpha}) \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ \mathbf{N}\mathfrak{a} \equiv 1 \bmod p^\beta}} R_\alpha(\mathfrak{a}) \lambda_\pi(\mathbf{N}_{K/F}(\mathfrak{a})) V \left(\frac{\mathbf{N}\mathfrak{a}}{Mp^{2(a+b)}} \right) \mathbf{N}\mathfrak{a}^{-\frac{1}{2}}. \end{aligned}$$

Here, $R_\alpha(\mathfrak{a})$ denotes the number of nonzero ideals $\mathfrak{a} \subset \mathcal{O}_K$ whose image in $\text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})$ lies in the trivial class. Also, we have used χ -orthogonality to obtain the first equality, and ρ -orthogonality to obtain the second. Writing this out in the Dirichlet series expansion (5) over nonzero ideals of \mathcal{O}_F then gives us $D(c, q) = D(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$.

We can evaluate the contribution of S_2 in a completely analogous fashion to obtain the stated contribution.

To evaluate the contributions from S_3 and S_4 , we must first take into account the contribution from summing over root numbers $\epsilon(1/2, \pi \times \rho\chi \circ \mathbf{N})$, using the formula (4) stated above. Thus let us first consider the sum over $\xi = \chi \circ \mathbf{N}$ of the Gauss sum term $\tau(\eta\xi^2) = \tau(\eta(\chi \circ \mathbf{N})^2)$ appearing in (4), writing $\mathbf{e}(x)$ for simplicity to denote the function $\exp(\mathrm{Tr}(xy)2\pi i)$. Again, y is our fixed class $y \in c(\eta\xi^2)\mathfrak{d}_F^{-1} \subset \mathcal{O}_F$. Recall too that we write $Q = \mathfrak{D}q$ to denote the conductor of $\eta\xi^2 = \eta(\chi \circ \mathbf{N})^2$. We have that

$$\begin{aligned} & \sum_{\chi \bmod p^\beta} \tau(\eta(\chi \circ \mathbf{N})^2)^4 \\ &= \sum_{\chi \bmod p^\beta} \sum_{x_1, x_2, x_3, x_4 \bmod Q} \eta(\chi \circ \mathbf{N})^2(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4) \\ &= \sum_{x_1, x_2, x_3, x_4 \bmod Q} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4) \sum_{\chi \bmod q} (\chi \circ \mathbf{N})^2(x_1 x_2 x_3 x_4) \\ &= \varphi(p^\beta) \sum_{\substack{x_1, x_2, x_3, x_4 \bmod Q \\ x_1^2 x_2^2 x_3^2 x_4^2 \equiv 1 \bmod q}} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4). \end{aligned}$$

Here, we have used the definition to obtain the first equality, interchanged summation to obtain the second, extracted invariant terms to obtain the third, and then evaluated via orthogonality to obtain the fourth. Thus, we have derived the formula (14)

$$\sum_{\chi \bmod p^\beta} \tau(\eta(\chi \circ \mathbf{N})^2)^4 = \varphi(p^\beta) \sum_{\substack{x_1, x_2, x_3, x_4 \bmod Q \\ x_1^2 x_2^2 x_3^2 x_4^2 \equiv 1 \bmod q}} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4).$$

We can use this to evaluate the contribution from the third sum S_3 as follows. Let us first write out the definition of this sum in the Dirichlet series expansion $\sum_{\mathfrak{a} \subset \mathcal{O}_K} \lambda_\pi \circ \mathbf{N}_{K/F}(\mathfrak{a}) \rho\chi \circ \mathbf{N}(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-s}$ over nonzero ideals of \mathcal{O}_K , which gives us

$$\begin{aligned} & \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\lambda_\pi(\mathbf{N}_{K/F}(\mathfrak{a}))}{\mathbf{N}\mathfrak{a}^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{a}}{Mp^{2(a+b)}}\right) \\ & \times \sum_{\rho \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})^\vee} \sum_{\chi \bmod p^\beta} \epsilon(1/2, \pi \times \rho(\chi \circ \mathbf{N})^2) \rho(\mathfrak{a})(\chi \circ \mathbf{N})(\mathfrak{a}) \end{aligned}$$

after extracting out the invariant terms for the sums over characters. Taking the description of the root number given in (4), written out in our notations here as

$$(15) \quad \epsilon(1/2, \pi \times \rho(\chi \circ \mathbf{N})) = \omega(c(\rho(\chi \circ \mathbf{N})^2)) \eta(\chi \circ \mathbf{N})^2(c(\pi)) \epsilon(1/2, \pi) \cdot \frac{\tau(\eta(\chi \circ \mathbf{N})^2)^4}{\mathbf{N}c(\eta(\chi \circ \mathbf{N})^2)^2},$$

we see that

$$\begin{aligned} & \sum_{\chi \bmod p^\beta} \epsilon(1/2, \pi \times \rho(\chi \circ \mathbf{N})) \\ &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2)) \eta(N) \epsilon(1/2, \pi)}{\mathbf{N}c(\eta(\chi \circ \mathbf{N})^2)^2} \sum_{\chi \bmod p^\beta} (\chi \circ \mathbf{N})^2(c(\pi)) \cdot \tau(\eta(\chi \circ \mathbf{N})^2)^4. \end{aligned}$$

Thus, writing $N = c(\pi) \subset \mathcal{O}_F$ for simplicity, we see that

$$\begin{aligned} & \sum_{\chi \bmod p^\beta} \epsilon(1/2, \pi \times \rho(\chi \circ \mathbf{N})^2) \\ &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)}{\mathbf{N}Q^2} \sum_{\chi \bmod p^\beta} (\chi \circ \mathbf{N})^2(N) \cdot \tau(\eta(\chi \circ \mathbf{N})^2)^4. \end{aligned}$$

This latter sum can be evaluated in a manner identical as for (14) to give us

$$\begin{aligned} & \sum_{\chi \bmod p^\beta} (\chi \circ \mathbf{N})^2(N) \cdot \tau(\eta(\chi \circ \mathbf{N})^2)^4 \\ &= \sum_{\chi \bmod p^\beta} \sum_{x_1, x_2, x_3, x_4 \bmod Q} \eta(x_1 x_2 x_3 x_4) (\chi \circ \mathbf{N})^2(N x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4) \\ &= \sum_{x_1, x_2, x_3, x_4 \bmod Q} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4) \sum_{\chi \bmod p^\beta} (\chi \circ \mathbf{N})^2(N x_1 x_2 x_3 x_4) \\ &= \sum_{\substack{x_1, x_2, x_3, x_4 \bmod Q \\ N^2 x_1^2 x_2^2 x_3^2 x_4^2 \equiv 1 \bmod q}} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4). \end{aligned}$$

Therefore, the sum over root numbers $\sum_{\chi \bmod p^\beta} \epsilon(1/2, \pi \rho(\chi \circ \mathbf{N}))$ is equal to

$$(16) \quad \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)}{\mathbf{N}Q^2} \sum_{\substack{x_1, x_2, x_3, x_4 \bmod Q \\ N^2 x_1^2 x_2^2 x_3^2 x_4^2 \equiv 1 \bmod q}} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4).$$

Hence the contribution from S_3 , again written as a sum over ideals of \mathcal{O}_K , equals

$$\begin{aligned} S_3 &= \sum_{\mathfrak{a} \subset \mathcal{O}_K} \lambda_\pi \circ \mathbf{N}_{K/F}(\mathfrak{a}) V \left(\frac{\mathbf{N}\mathfrak{a}^{\frac{1}{2}}}{Mp^{2(a+b)}} \right) \mathbf{N}\mathfrak{a}^{-\frac{1}{2}} \\ &\times \sum_{\rho \in \text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})^\vee} \rho^{-1}(\mathfrak{a}) \sum_{\chi \bmod p^\beta} (\chi^{-1} \circ \mathbf{N})^2(\mathfrak{a}) \epsilon(1/2, \pi \times \rho(\chi \circ \mathbf{N})^2). \\ &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)}{\mathbf{N}Q^2} \sum_{\mathfrak{a} \subset \mathcal{O}_K} \lambda_\pi \circ \mathbf{N}_{K/F}(\mathfrak{a}) R_\alpha(\mathfrak{a}) V \left(\frac{\mathbf{N}\mathfrak{a}^{\frac{1}{2}}}{Mp^{2(a+b)}} \right) \mathbf{N}\mathfrak{a}^{-\frac{1}{2}} \\ &\times h(\mathcal{O}_{\mathfrak{p}^\alpha}) \sum_{\chi \bmod p^\beta} (\chi^{-1} \circ \mathbf{N})^2(\mathfrak{a}) \cdot \tau(\eta(\chi \circ \mathbf{N})^2)^4. \end{aligned}$$

Here, we have used orthogonality to evaluate the second sum over ring class characters of conductor \mathfrak{p}^α . Using this latter expression, it now follows from (16) that

$$\begin{aligned} S_3 &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)}{\mathbf{N}Q^2} \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\lambda_\pi \circ \mathbf{N}_{K/F}(\mathfrak{a}) R_\alpha(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^{\frac{1}{2}}} V \left(\frac{\mathbf{N}\mathfrak{a}^{\frac{1}{2}}}{Mp^{2(a+b)}} \right) \\ &\times h(\mathcal{O}_{\mathfrak{p}^\alpha}) \varphi(\mathfrak{p}^\beta) \sum_{\substack{x_1, x_2, x_3, x_4 \bmod Q \\ \mathbf{N}_{K/F}(\mathfrak{a}) N^2 x_1^2 x_2^2 x_3^2 x_4^2 \equiv 1 \bmod q}} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4), \end{aligned}$$

Thus, writing out the expansion of the latter expression over ideals of \mathcal{O}_F gives us the stated contribution $D^*(c, q) = D^*(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$. Again, a completely analogous

argument can be used to evaluate the coefficients in the fourth sum S_4 to obtain the stated contribution $\mathfrak{E}^*(c, q) = \mathfrak{E}^*(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$.

Finally, the arguments for the single averages (8) work in completely the same way after fixing the ring class character ρ . Since the arguments essentially are identical up to choice of notation, we omit them for brevity. \square

Lemma 4.3. *We have (for any choice of ring class conductor $c = c(\rho) = \mathfrak{p}^\alpha$) that $\lim_{q \rightarrow \infty} \mathfrak{E}(c, q) = \lim_{q \rightarrow \infty} \mathfrak{E}^*(c, q) = \lim_{q \rightarrow \infty} \mathfrak{E}(q) = \lim_{q \rightarrow \infty} \mathfrak{E}^*(q) = 0$.*

Proof. See the argument of [30, Proposition 3.5]. The same argument works for each of the stated cases here, as the residues at $s = 0$ coming from the modified cutoff functions \mathfrak{V}_j vanish thanks to the appearance of the factors $(1 - \mathbf{N}\mathfrak{p}^{-2s})$. \square

4.2. Estimates. We now proceed to estimates. We shall treat both averages (7) and (8) along the same lines in the discussion that follows, the key being to evaluate the off-diagonal terms $D^*(c, q)$ and $D^*(q)$ respectively. We shall also assume for simplicity of exposition that the relative discriminant $\mathfrak{D} = \mathfrak{D}_{K/F}$ is prime to \mathfrak{p} in the discussion that follows.

Let us begin with the double averages $H(c, q)$ defined above in (7). The results of Theorem 4.2 and Lemma 4.5 indicate that it will suffice to estimate the sum $D(c, q) + D^*(c, q)$ as $q \rightarrow \infty$ for our main averaging theorem. Let us first consider

$$\begin{aligned} D^*(c, q) &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)}{\mathbf{N}Q^2} \\ &\times \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \subset \mathcal{O}_F} \frac{\lambda_\pi(\mathfrak{n})r_\alpha(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}}\right) \\ &\times \sum_{\substack{x_1, x_2, x_3, x_4 \bmod Q \\ \mathfrak{m}^2 \mathfrak{n} N^2 x_1^2 x_2^2 x_3^2 x_4^2 \equiv 1 \bmod q}} \eta(x_1 x_2 x_3 x_4) \mathbf{e}(x_1 + x_2 + x_3 + x_4), \end{aligned}$$

where we have kept all of the notations and conventions from the discussion above. Thus for instance, we have written $Q = \mathfrak{D}q$ to denote the conductor $c(\eta\xi) \subset \mathcal{O}_F$ of the idele class character $\eta\xi$ of F , with $\mathfrak{D} = \mathfrak{D}_{K/F}$ the conductor of $\eta = \eta_{K/F}$, which we assume for simplicity to be coprime to $q = \mathfrak{p}^\beta$. To simplify this expression, let us first consider that since \mathfrak{D} and q are coprime, we have for any class $P \bmod q$ the decomposition

$$\begin{aligned} &\sum_{\substack{x \bmod \mathfrak{D}q \\ x^2 \equiv P \bmod q}} \eta(x) \exp(\mathrm{Tr}(xy)2\pi i) \\ &= \sum_{w \bmod \mathfrak{D}} \eta(w) \exp(\mathrm{Tr}(wy_{\mathfrak{D}})2\pi i) \sum_{\substack{z \bmod q \\ z \equiv \pm J_P \bmod q}} \exp(\mathrm{Tr}(zy_q)2\pi i). \end{aligned}$$

Here, $y_{\mathfrak{D}}$ denotes the induced representative of $\mathfrak{D}\mathfrak{d}^{-1}$, and y_q that of $q\mathfrak{d}^{-1}$, so that $\mathrm{Tr}(xy) = \mathrm{Tr}(wy_{\mathfrak{D}}) + \mathrm{Tr}(zy_q)$ for $x = wz$ a class of $Q = \mathfrak{D}q$, whence

$$\exp(\mathrm{Tr}(xy)2\pi i) = \exp(\mathrm{Tr}(wy_{\mathfrak{D}})2\pi i) \exp(\mathrm{Tr}(zy_q)2\pi i).$$

We have also written J_P to denote a square root of $P \bmod q$ if it exists. That is, we use the Chinese remainder theorem (cf. [30, Corollary 3.10]) to argue that

$$\begin{aligned} & \sum_{\substack{x \bmod \mathfrak{D}q \\ x^2 \equiv P \bmod q}} \eta(x) \exp(\mathrm{Tr}(xy)2\pi i) \\ &= \sum_{w \bmod \mathfrak{D}} \eta(w) \sum_{\substack{z \bmod q \\ zw \equiv \pm J_P \bmod \mathfrak{D}q}} \exp(\mathrm{Tr}(wy_{\mathfrak{D}})2\pi i) \exp(\mathrm{Tr}(zy_q)2\pi i). \end{aligned}$$

Hence, by the definition of the Gauss sum $\tau(\eta)$, we derive that

$$(17) \quad \sum_{\substack{x \bmod \mathfrak{D}q \\ x^2 \equiv P \bmod q}} \eta(x) \exp(\mathrm{Tr}(xy)2\pi i) = \tau(\eta) \sum_{\substack{z \bmod q \\ zw \equiv \pm J_P \bmod \mathfrak{D}q}} \exp(\mathrm{Tr}(zy_q)2\pi i)$$

Using this expression (17) four times in our formula for $D^*(c, q)$, and writing $\mathfrak{K}_4(J_P; q)$ to denote the hyper-Kloosterman style sum defined by

$$(18) \quad \mathfrak{K}_4(J_P; q) = \sum_{\substack{z_1, z_2, z_3, z_4 \bmod q \\ z_1 z_2 z_3 z_4 \equiv J_P \bmod q}} \exp(\mathrm{Tr}(z_1 y_q + z_2 y_q + z_3 y_q + z_4 y_q)2\pi i),$$

we find that

$$\begin{aligned} D^*(c, q) &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)\tau(\eta)^4}{\mathbf{N}\mathfrak{D}^2\mathbf{N}q^2} \\ &\times \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \subset \mathcal{O}_F} \frac{\lambda_{\pi}(\mathfrak{n})r_{\alpha}(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}}\right) \\ &\times \{\mathfrak{K}_4(J_{\mathfrak{m}, \mathfrak{n}, N}; q) + \mathfrak{K}_4(-J_{\mathfrak{m}, \mathfrak{n}, N}; q)\}, \end{aligned}$$

where $J_{\mathfrak{m}, \mathfrak{n}, N}$ denotes a (choice of) square root of the inverse class of $\mathfrak{m}^2\mathbf{N}^2 \bmod q$ if it exists. A completely analogous argument shows that, for ρ a fixed ring class character of conductor $c = c(\rho) = \mathfrak{p}^{\alpha}$, the corresponding sum $D^*(q) = D_{\rho}^*(q)$ equals

$$\begin{aligned} D^*(q) &= \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)\tau(\eta)^4}{\mathbf{N}\mathfrak{D}^2\mathbf{N}q^2} \\ &\times \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \subset \mathcal{O}_F} \frac{\lambda_{\pi}(\mathfrak{n})r_{\rho}(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}}\right) \\ &\times \{\mathfrak{K}_4(J_{\mathfrak{m}, \mathfrak{n}, N}; q) + \mathfrak{K}_4(-J_{\mathfrak{m}, \mathfrak{n}, N}; q)\}. \end{aligned}$$

Proposition 4.4. *Recall that we fix a representative $y \in \mathfrak{p}^{\beta}\mathfrak{d}_F^{-1} \subset \mathcal{O}_F$. If a class J modulo $q = \mathfrak{p}^{\beta}$ admits a fourth root of unity $J^{1/4}$ modulo q , then we have that*

$$\mathfrak{K}_4(J; q) + \mathfrak{K}_4(-J; q) = 2(\mathbf{N}\mathfrak{p}^{\beta})^{\frac{3}{2}} \sum_{\zeta^4 \equiv 1 \bmod \mathfrak{p}^{\beta}} \cos\left(2\pi \cdot \mathrm{Tr}(4J^{1/4}\zeta y)\right).$$

Here, the sum runs over roots of unity ζ in $(\mathcal{O}_F/\mathfrak{p}^{\beta}\mathcal{O}_F)^{\times}$.

Proof. The result follows from a direct calculation in the style of Salié, in the same way as for the classical case (cf. [30, Proposition 3.11]). In particular, we can use the same general argument given in Blomer-Brumley [2, (C.5)] to show that

$$(19) \quad \mathfrak{K}_4(J; \mathfrak{p}^{\beta}) = (\mathbf{N}\mathfrak{p}^{\beta})^{\frac{3}{2}} \sum_{\zeta^4 \equiv 1 \bmod \mathfrak{p}^{\beta}} \exp(\mathrm{Tr}(4J^{1/4}\zeta)2\pi i).$$

That is, the argument given for [2, Lemme C.4] extends directly after replacing their $f(\mathbf{x}) = x_1 + x_2 + x_3 + x_4$ with the function $g(\mathbf{x}) = \text{Tr}(y(x_1 + x_2 + x_3 + x_4))$ obtained by composing with multiplication by y and the trace homomorphism Tr . Since $g = f \circ \text{Tr} \circ y$ is obtained from f by linear transformation, we claim it is easy to check that the calculations for [2, Lemme C.4] used to prove [2, (C.5)] carry over directly. Granted (19), the result is easy to derive after using the definition $\exp(w) = \cos(w) + i \sin(w)$ of the exponential function on $w \in i\mathbf{R}$. \square

Proposition 4.5. *Let $\theta \in [0, 1/2]$ denote the best bound known towards the Ramanujan-Selberg conjecture over number fields, which thanks to Blomer-Brumley [3] we can take to be $\theta = 7/64$. We have for any choice of $\varepsilon > 0$ that*

$$D^*(c, q) = O_{p, \pi, \mathfrak{D}, c, \varepsilon} \left(\mathbf{N}q^{2\theta - \frac{3}{2} + \varepsilon} \log \mathbf{N}q \right),$$

and similarly that

$$D^*(q) = O_{p, \pi, \mathfrak{D}, \rho, \varepsilon} \left(\mathbf{N}q^{2\theta - \frac{3}{2} + \varepsilon} \log \mathbf{N}q \right).$$

Proof. Cf. [30, Proposition 3.11]. We first consider the largest possible contribution to the sum $D^*(c, q)$ coming from a pair of nonzero ideals $(\mathfrak{m}, \mathfrak{n})$ of \mathcal{O}_F . Using the rapid decay of the cutoff function V (Lemma 3.1 for the region $y \geq 1$), we can assume without loss of generality that $\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n} \leq Mp^{2(a+b)}$. We then deduce from the discussion given above that such a pair $(\mathfrak{m}, \mathfrak{n})$ contributes to the sum only if (i) the inverse class of $\mathfrak{m}^2\mathfrak{n}N^2$ modulo q admits a square root $J_{\mathfrak{m}, \mathfrak{n}, N}$ and (ii) the class $J_{\mathfrak{m}, \mathfrak{n}, N} \bmod q$ admits of fourth root $J_{\mathfrak{m}, \mathfrak{n}, N}^{1/4}$ modulo q . When both conditions (i) and (ii) are satisfied, we obtain an exact contribution of

$$\begin{aligned} & \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)\tau(\eta)^4}{\mathbf{N}\mathfrak{D}^2\mathbf{N}q^2} \cdot V\left(\frac{\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}}\right) \frac{\omega\eta(\mathfrak{m})\lambda_\pi(\mathfrak{n})r_\alpha(\mathfrak{n})}{\mathbf{N}\mathfrak{m}\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} \\ & \times \mathbf{N}q^{\frac{3}{2}} \sum_{\zeta^4 \equiv 1 \bmod q} 2 \cos\left(2\pi \cdot \text{Tr}(4J_{\mathfrak{m}, \mathfrak{n}, N}^{1/4}\zeta)\right), \end{aligned}$$

to $D^*(c, q)$, which is the same as

$$\begin{aligned} & \frac{\omega(c(\rho(\chi \circ \mathbf{N})^2))\eta(N)\epsilon(1/2, \pi)\tau(\eta)^4}{\mathbf{N}\mathfrak{D}^2\mathbf{N}q^{\frac{1}{2}}} \cdot V\left(\frac{\mathbf{N}\mathfrak{m}^2\mathbf{N}\mathfrak{n}}{Mp^{2(a+b)}}\right) \frac{\omega\eta(\mathfrak{m})\lambda_\pi(\mathfrak{n})r_\alpha(\mathfrak{n})}{\mathbf{N}\mathfrak{m}\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} \\ & \sum_{\zeta^4 \equiv 1 \bmod q} 2 \cos\left(2\pi \cdot \text{Tr}(4J_{\mathfrak{m}, \mathfrak{n}, N}^{1/4}\zeta)\right). \end{aligned}$$

Now, recall that $q = \mathfrak{p}^\beta$ has norm $\mathbf{N}q = p^{\delta\beta} = p^b$, whence the condition in the ζ -sum is equivalent to having a solution to $\mathbf{N}\zeta \equiv 1 \bmod p^b$. Thus by Hensel's lemma, the contribution of the ζ -sum to $D^*(c, q)$ depends only upon the roots of $h(x) = x^4 - 1 \bmod p$. That is, the ζ -sum contributes only $O_p(1)$. Recall as well that by Lemma 3.1, we have the bound $V(Y) = O(Y^{\frac{1}{2}})$ for $0 < Y \leq 1$. Putting this together with the classical bound $r_\alpha(\mathfrak{n}) = O_\varepsilon(\mathbf{N}\mathfrak{n}^\varepsilon)$ and the best known bound towards the Ramanujan-Selberg conjecture $\lambda_\pi(\mathfrak{n}) = O_\varepsilon(\mathbf{N}\mathfrak{n}^{\theta+\varepsilon})$ then allows us to bound this contribution trivially. That is, we see that for any choices of $\varepsilon_1, \varepsilon_2 > 0$, the contribution of the pair $(\mathfrak{m}, \mathfrak{n})$ is bounded by

$$O_{p, \varepsilon_1, \varepsilon_2} \left(\frac{\mathbf{N}\mathfrak{n}^{\theta+\varepsilon_1+\varepsilon_2}}{M^{\frac{1}{2}}p^a\mathbf{N}q^{\frac{3}{2}}} \right) = O_{p, \pi, \mathfrak{D}, c, \varepsilon_1, \varepsilon_2} \left(\frac{\mathbf{N}\mathfrak{n}^{\theta+\varepsilon_1+\varepsilon_2}}{\mathbf{N}q^{\frac{3}{2}}} \right).$$

Note that this bound is uniform in \mathfrak{m} , and moreover that the largest \mathfrak{n} in the region $\mathbf{Nm}^2\mathbf{Nn} \leq Mp^{2(a+b)}$ which can contribute nontrivially to $D^*(c, q)$ is subject to the constraint $\mathbf{Nn} \leq Mp^{2(a+b)}\mathbf{Nm}^{-2} = O_{\pi, \mathfrak{D}, c}(\mathbf{N}q^2)$. On the other hand, notice that by Hensel's lemma, the constraints (i) and (ii) can be replaced by their mod p analogues. That is, a pair of nonzero ideals $(\mathfrak{m}, \mathfrak{n})$ gives a nonzero contribution to the sum $D^*(c, q)$ only if (iii) there exists a square root $Q_{\mathfrak{m}, \mathfrak{n}, N}$ of the inverse class of $\mathbf{Nm}^2\mathbf{NnNN} \bmod p$ and (iv) there exists a fourth root $Q_{\mathfrak{m}, \mathfrak{n}, N}^{1/4}$ of this class $Q_{\mathfrak{m}, \mathfrak{n}, N} \bmod p$. Hence, writing $W = W_{8, p}$ to denote the number of octic residues mod p , we have at most W classes in $(\mathbf{Z}/p\mathbf{Z})^\times$ which satisfy both constraints (iii) and (iv). It follows that we have at most $4bW = O_p(\log \mathbf{N}q)$ possible ideals \mathfrak{n} in the region defined by $\mathbf{Nn} \leq Mp^{2(a+b)}\mathbf{Nm}^{-2}$ to consider. In other words, it follows that we have at most $4bW = O_p(\log \mathbf{N}q)$ pairs $(\mathfrak{m}, \mathfrak{n})$ to consider along the hyperbola region defined by $\mathbf{Nm}^2\mathbf{Nn} \leq Mp^{2(a+b)}$, each one contributing at most $O_{p, \pi, \mathfrak{D}, c, \varepsilon_1, \varepsilon_2}(\mathbf{N}q^{2\theta - \frac{3}{2} + 2(\varepsilon_1 + \varepsilon_2)})$. Therefore, choosing $\varepsilon_1, \varepsilon_2 > 0$ in such a way that $\varepsilon = 2(\varepsilon_1 + \varepsilon_2)$, we have that

$$D^*(c, q) = O_{p, \pi, \mathfrak{D}, c, \varepsilon}(\mathbf{N}q^{2\theta - \frac{3}{2} + \varepsilon} \log \mathbf{N}q).$$

An identical argument works to show that $D^*(q) = O_{p, \pi, \mathfrak{D}, \rho, \varepsilon}(\mathbf{N}q^{2\theta - \frac{3}{2} + \varepsilon} \log \mathbf{N}q)$ after using the trivial bound $r_\rho(\mathfrak{n}) \ll r_\alpha(\mathfrak{n})$ for any nonzero ideal $\mathfrak{n} \subset \mathcal{O}_F$. \square

Putting these results together, we obtain the following.

Theorem 4.6. *There exist absolute constants $\kappa_1, \kappa_2 > 0$ such that for all $q = \mathfrak{p}^\beta$ of sufficiently large norm $\mathbf{N}q = p^{\beta\delta} = p^b$ (i) the average (7) is given by the formula*

$$H(c, q) = \varphi(p^\beta)^{-1} \sum_{\xi \bmod \mathfrak{p}^\beta} L(1, \omega \eta \xi^2) + O_{p, \pi, \mathfrak{D}, c, \varepsilon}(\mathbf{N}q^{2\theta - \frac{3}{2} + \varepsilon - \kappa_1} \log \mathbf{N}q)$$

and (ii) the average (8) by the formula

$$H(q) = \varphi(\mathfrak{p}^\beta)^{-1} \sum_{\xi \bmod \mathfrak{p}^\beta} L(1, \omega \eta \xi^2) + O_{p, \pi, \mathfrak{D}, \rho, \varepsilon}(\mathbf{N}q^{2\theta - \frac{3}{2} + \varepsilon - \kappa_2} \log \mathbf{N}q).$$

Proof. Cf. [30, Proposition 3.12]. Let us start with the average $H(c, q)$. We deduce from Proposition 4.5 that for β sufficiently large the average is given up to bounded error term $D^*(c, q) = O_{p, \pi, \mathfrak{D}, c, \varepsilon}(\mathbf{N}q^{2\theta - \frac{3}{2} + 2\varepsilon} \log \mathbf{N}q)$ by the sum

$$D(c, q) = \varphi(\mathfrak{p}^\beta)^{-1} \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\omega \eta(\mathfrak{m})}{\mathbf{Nm}} \sum_{\substack{\mathfrak{n} \subset \mathcal{O}_F \\ \mathfrak{m}^2 \mathfrak{n} \equiv 1 \bmod \mathfrak{p}^\beta}} \frac{\lambda_\pi(\mathfrak{n}) r_\alpha(\mathfrak{n})}{\mathbf{Nn}^{\frac{1}{2}}} V\left(\frac{\mathbf{Nm}^2 \mathbf{Nn}}{Mp^{2(a+b)}}\right).$$

We can again assume without loss of generality that $\mathbf{Nm}^2\mathbf{Nn} \leq Mp^{2(a+b)}$ by the rapid decay of the cutoff function V (Lemma 3.1). This reduces us to summing over pairs $(\mathfrak{m}, \mathfrak{n})$ in this region subject to the congruence condition $\mathfrak{m}^2\mathfrak{n} \equiv 1 \bmod q$. On the other hand, recall that $p^{2(a+b)} = \mathbf{N}c^2\mathbf{N}q^2$. Notice as well that we only sum over ideals which are prime to the conductor $cq = \mathfrak{p}^{\alpha+\beta}$. We claim it is then easy to see for β sufficiently large (e.g. $p^{\beta\delta} \gg Mp^{2a}$) that the only non-negligible solution to the congruence $\mathfrak{m}^2\mathfrak{n} \equiv 1 \bmod q$ will be given by $\mathfrak{m}^2 = \mathfrak{n} = 1$. This means for instance that we are reduced to looking at the sum over $\mathfrak{m}^2 \equiv 1 \bmod q$ to estimate

$D(c, q)$, although the \mathfrak{m} of large norm here will contribute only negligibly thanks to the rapid decay of V . Thus, we reduce to looking at the sum

$$\sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m}^2 \equiv 1 \pmod{q}}} \frac{\omega\eta(\mathfrak{m})}{\mathbf{Nm}} V\left(\frac{\mathbf{Nm}^2}{Mp^{2(a+b)}}\right) = \int_{\Re(s)=2} \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m}^2 \equiv 1 \pmod{q}}} \frac{\omega\eta(\mathfrak{m})}{\mathbf{Nm}^{2s+1}} \widehat{V}(s) \left(Mp^{2(a+b)}\right)^s \frac{ds}{2\pi i},$$

which is the same as the integral

$$\begin{aligned} & \int_{\Re(s)=2} \varphi(p^\beta)^{-1} \sum_{\chi \pmod{p^\beta}} L(2s+1, \omega\eta(\chi \circ \mathbf{N})^2) \widehat{V}(s) \left(Mp^{2(a+b)}\right)^s \frac{ds}{2\pi i} \\ &= \varphi(p^\beta)^{-1} \sum_{\chi \pmod{p^\beta}} \int_{\Re(s)=2} L(2s+1, \omega\eta(\chi \circ \mathbf{N})^2) \widehat{V}(s) \left(Mp^{2(a+b)}\right)^s \frac{ds}{2\pi i}. \end{aligned}$$

The result can now be derived from a standard residue computation, as given in [30, Lemma 3.7 and Proposition 3.12]. Briefly, one moves the line of integration to $\Re(s) = -1/8$, crossing a pole at $s = 0$ of residue

$$\varphi(p^\beta)^{-1} \sum_{\chi \pmod{p^\beta}} L(1, \omega\eta(\chi \circ \mathbf{N})^2) = \varphi(p^\beta)^{-1} \sum_{\xi \pmod{\mathfrak{p}^\beta}} L(1, \omega\eta\xi^2).$$

The remaining contribution can then be bounded using the rapid decay of the cutoff function V as $\Im(s) \rightarrow \pm\infty$, along with Burgess subconvexity bounds (given in [4]). This allows us to deduce the claim for the double average $H(c, q)$. A completely analogous argument works for the single averages granted Proposition 4.5, replacing the coefficients $r_\alpha(\mathfrak{n})$ in $H(c, q)$ with the coefficients $r_\rho(\mathfrak{n})$ in $H(q)$. \square

Corollary 4.7. (i) If the exponent $\beta \geq 0$ is sufficiently large, then for any fixed choice of $\alpha \geq 0$, the average $H(c, q) = H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = H_{\pi, \mathfrak{D}}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ does not vanish. (ii) If the exponent $\beta \geq 0$ is sufficiently large, then for any choice of ring class character $\rho \in X(\mathfrak{p})$, the average $H(q) = H(\mathfrak{p}^\beta) = H_{\pi, \mathfrak{D}, \rho}(\mathfrak{p}^\beta)$ does not vanish.

Proof. Consider first the average $H(c, q)$ after the result of Theorem 4.6. Fix a real number $X \geq 1$. Applying partial summation to each of the Dirichlet series $L(s, \omega\eta\xi^2)$, and using the Pólya-Vinogradov for the error term, we obtain that

$$L(s, \omega\eta\xi^2) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathbf{Nm} \leq X}} \frac{\omega\eta\xi^2(\mathfrak{m})}{\mathbf{Nm}} + O_{\mathfrak{D}}\left(\mathbf{N}q^{\frac{1}{2}} \log \mathbf{N}q X^{-\Re(s)}\right).$$

Taking $s = 1$, then summing over characters $\xi = \chi \circ \mathbf{N} \pmod{q = \mathfrak{p}^\beta}$, we have

$$\varphi(\mathfrak{p}^\beta)^{-1} \sum_{\xi \pmod{\mathfrak{p}^\beta}} L(1, \omega\eta\xi^2) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m}^2 \equiv 1 \pmod{\mathfrak{p}^\beta}}} \frac{\omega\eta(\mathfrak{m})}{\mathbf{Nm}} + O_{\mathfrak{D}}\left(\mathbf{N}q^{\frac{1}{2}} \log \mathbf{N}q X^{-1}\right).$$

Taking $X = \mathbf{N}q$ then gives us the estimate

$$\varphi(\mathfrak{p}^\beta)^{-1} \sum_{\xi \pmod{\mathfrak{p}^\beta}} L(1, \omega\eta\xi^2) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m}^2 \equiv 1 \pmod{\mathfrak{p}^\beta}}} \frac{\omega\eta(\mathfrak{m})}{\mathbf{Nm}} + O_{\mathfrak{D}}\left(\mathbf{N}q^{-\kappa}\right)$$

for some constant $\kappa > 0$. This proves the claim for the double average $H(c, q)$. The same argument works for the single average $H(q)$ after invoking Theorem 4.6. \square

5. SOME REFINEMENTS

We now present some refinements of the result of Corollary 4.7. We first indicate how the Möbius inversion formula can be used to get information about sets of primitive characters. We then recall the algebraicity theorem of Shimura [28], and show how to derive results about the associated (cyclotomic) Galois averages in special cases. Finally, we describe the existence of a related p -adic L -function (in particular that given by Hida [10]), and show how this can also be used in a completely soft way to obtain nonvanishing results for Galois conjugate families. Thus, we give two methods of establishing the nontriviality of these Galois averages.

5.1. Primitive averages. Note that the averages $H(c, q) = H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ of (7) and $H(q) = H(\mathfrak{p}^\beta)$ of (8) are not defined with respect to primitive sets of characters. We can define and estimate the analogous averages over primitive sets as follows. Recall that for arbitrary positive integers $\alpha, \beta \geq 0$, we write $P_{\alpha, \beta}$ to denote the set of characters $\mathcal{W} = \rho\psi = \rho\xi \circ \mathbf{N}_{K/F} = \rho\chi \circ \mathbf{N}$, where ρ is a primitive ring class character of conductor $c(\rho) = \mathfrak{p}^\alpha$, and $\xi = \chi \circ \mathbf{N}_{F/\mathbf{Q}}$ is a primitive cyclotomic character of conductor \mathfrak{p}^β (i.e. arising from a primitive Dirichlet character χ of conductor p^β). Recall as well that we write P_β to denote the set of primitive cyclotomic characters ξ of conductor \mathfrak{p}^β in this sense. We can then define the associated averages

$$\mathcal{P}(c, q) = \mathcal{P}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = \mathcal{P}_{\pi, \mathfrak{D}}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = |P_{\alpha, \beta}|^{-1} \sum_{\rho' \chi' \circ \mathbf{N} \in X_{\alpha, \beta}} L(1/2, \pi \times \rho' \chi' \circ \mathbf{N})$$

and (for ρ fixed)

$$\mathcal{P}(q) = \mathcal{P}(\mathfrak{p}^\beta) = \mathcal{P}_{\pi, \mathfrak{D}, \rho}(\mathfrak{p}^\beta) = |P_\beta|^{-1} \sum_{\chi' \circ \mathbf{N} \in X_\beta} L(1/2, \pi \times \rho' \chi' \circ \mathbf{N}).$$

Using the decomposition (10), and writing $h_{\alpha, \beta} = |\text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})| \varphi(p^\beta)$ to denote the cardinality of the full set of characters $X_{\alpha, \beta}$, we obtain from definitions that

$$(20) \quad H(c, q) = H(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = h_{\alpha, \beta}^{-1} \sum_{\substack{0 \leq x \leq \alpha \\ 0 \leq y \leq \beta}} |P_{x, y}| \cdot \mathcal{P}(\mathfrak{p}^x, \mathfrak{p}^y)$$

and

$$(21) \quad H(q) = H(\mathfrak{p}^\beta) = h_\beta^{-1} \sum_{0 \leq y \leq \beta} |P_y| \cdot \mathcal{P}(\mathfrak{p}^y).$$

Two applications of the Möbius inversion formula to (20) give us the relation

$$(22) \quad |P_{\alpha, \beta}| \cdot \mathcal{P}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = \sum_{\substack{0 \leq x \leq \alpha \\ 0 \leq y \leq \beta}} \mu\left(\frac{p^{\alpha\delta}}{p^{x\delta}}\right) \mu\left(\frac{p^\beta}{p^y}\right) h_{x, y} \cdot H(\mathfrak{p}^x, \mathfrak{p}^y),$$

and one application of it to (21) gives us the relation

$$(23) \quad |P_\beta| \cdot \mathcal{P}(\mathfrak{p}^\beta) = \sum_{0 \leq y \leq \beta} \mu\left(\frac{p^\beta}{p^y}\right) h_y \cdot H(\mathfrak{p}^y).$$

Now, it is not difficult to derive the following result from Corollary 4.7 above.

Proposition 5.1. (i) If the exponent $\beta \geq 0$ is sufficiently large, then the difference of sums (22) does not vanish. (ii) If the exponent $\beta \geq 0$ is sufficiently large, then the difference of sums (23) does not vanish.

Proof. See the argument of [30, Proposition 4.3] in the non self-dual setting, which carries over in the same way here. To be more precise, using that we only ever look at \mathfrak{p} -power conductors to simplify the expressions (22) and (23), along with Dedekind's formula (9) for the class number $h(\mathcal{O}_{\mathfrak{p}^\alpha})$ of $\text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha})$, and the result of Corollary 4.7 for input, the argument works in the same way. \square

Thus we obtain the following immediate consequence.

Corollary 5.2. (i) *If the exponent $\beta \geq 0$ is sufficiently large, then for any fixed choice of $\alpha \geq 0$, the primitive average $\mathcal{P}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta) = \mathcal{P}_{\pi, \mathfrak{D}}(\mathfrak{p}^\alpha, \mathfrak{p}^\beta)$ does not vanish.*
(ii) *If the exponent $\beta \geq 0$ is sufficiently large, then for any choice of ring class character $\rho \in X(\mathfrak{p})$, the primitive average $\mathcal{P}(\mathfrak{p}^\beta) = \mathcal{P}_{\pi, \mathfrak{D}, \rho}(\mathfrak{p}^\beta)$ does not vanish.*

5.2. Galois averages. We can now deduce from Corollary 5.2 the following consequence for certain Galois averages of the form defined in (2) above.

Corollary 5.3. *Assume that K/F is totally imaginary, and that π is a holomorphic discrete series of weight $(k_j)_{j=1}^d$ with each $k_j \geq 2$. Fix a ring class character $\rho \in X(\mathfrak{p})$ of K . Assume that the Hecke field $\mathbf{Q}(\pi)$ is linearly disjoint over \mathbf{Q} to the cyclotomic field $\mathbf{Q}(\rho)$, as well as to the cyclotomic tower $\mathbf{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbf{Q}(\zeta_{p^n})$. If the exponent $\beta \geq 0$ is sufficiently large, then for each character $\xi = \chi \circ \mathbf{N} \in P_\beta$, the Galois average $G_{[\rho\chi \circ \mathbf{N}]}(\pi)$ does not vanish. Thus for all but finitely many cyclotomic characters $\psi = \xi \circ \mathbf{N}_{K/F} = \chi \circ \mathbf{N}$ whose underlying Dirichlet character χ has p -power conductor, $L(1/2, \pi \times \rho\psi) = L(1/2, \pi \times \rho\xi \circ \mathbf{N}_{K/F}) = L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ does not vanish.*

Proof. The result of Corollary 5.2 (ii) above implies that for $\beta \gg 0$, the sum

$$\sum_{\xi = \chi \circ \mathbf{N}_{F/\mathbf{Q}} \in P_\beta} L(1/2, \pi \times \rho\chi \circ \mathbf{N})$$

does not vanish, whence the associated Galois average $G_{[\rho\chi \circ \mathbf{N}]}(\pi)$ cannot vanish. This is because each of the values $L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ with χ ranging over primitive Dirichlet characters mod p^β is contained in the sum (2) defining $G_{[\rho\chi \circ \mathbf{N}]}(\pi)$. Here, we have used implicitly the fact that the characters $\xi = \chi \circ \mathbf{N}$ in the set P_β are Galois conjugate, which is rather particular to this cyclotomic setting (and for instance not generally true of the characters contained in the larger sets $P_{\alpha, \beta}$). \square

Remark Note that we have strayed from detail on the subtler issue of averaging over Galois conjugate ring class characters ρ of K . This problem is addressed in the work [32] (cf. also Rohrlich [26] for the classical setting with CM elliptic curves), and also in the final subsection of this work (for a special case via basechange). In short, the ring class characters ρ of K of a given conductor \mathfrak{p}^α might have order which is not exactly equal to $\mathbf{N}\mathfrak{p}^\alpha = p^{\alpha\delta} = p^a$, and hence the set of primitive ring class characters of conductor \mathfrak{p}^α might in general contain several Galois orbits of ring class characters.

5.3. p -adic L -functions. We now explain how stronger nonvanishing results can be deduced from Corollary 4.7 when we assume (i) K/F is totally imaginary and (ii) π is a holomorphic discrete series of weight $(k_j)_{j=1}^d$ with each $k_j \geq 2$ (i.e. with weights strictly greater than those of the representations $\pi(\mathcal{W})$) from the existence of a suitable p -adic L -function (e.g. by Hida [10]). We shall also assume for simplicity of exposition that (iii) π is \mathfrak{p} -ordinary, i.e. the image of the eigenvalue of the Hecke operator at \mathfrak{p} of our fixed embedding $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ is a p -adic unit.

5.3.1. *Iwasawa algebras and formal power series rings.* Let \mathcal{O} be a finite extension of \mathbf{Z}_p , and \mathcal{G} a profinite group. We write

$$\mathcal{O}[[\mathcal{G}]] = \varprojlim_{\mathcal{U} \subset \mathcal{G}} \mathcal{O}[\mathcal{G}/\mathcal{U}]$$

to denote the \mathcal{O} -Iwasawa algebra of \mathcal{G} , equivalently the completed group ring of \mathcal{G} with coefficients in \mathcal{O} . Hence, the limit runs over open normal subgroups $\mathcal{U} \subset \mathcal{G}$, and the elements \mathcal{L} of $\mathcal{O}[[\mathcal{G}]]$ can be viewed as \mathcal{O} -valued measures $d\mathcal{L}$ on \mathcal{G} .

We shall consider the following choice of profinite group \mathcal{G} in our discussion. Essentially, we shall take $\mathcal{G} = \varprojlim_{\alpha, \beta} \text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha}) \times (\mathcal{O}_F/\mathfrak{p}^\beta \mathcal{O}_F)^\times$. To be consistent with the literature however, let us give the following equivalent Galois theoretic definition. That is, composing with the reciprocity map rec_K of class field theory for K gives us identifications $\text{rec}_K : \text{Pic}(\mathcal{O}_{\mathfrak{p}^\alpha}) \cong \Omega_\alpha = \text{Gal}(K[\mathfrak{p}^\alpha]/K)$ for each exponent $\alpha \geq 0$, i.e. where $K[\mathfrak{p}^\alpha]$ denotes the (finite abelian) ring class extension of K of conductor \mathfrak{p}^α . The torsion subgroup Ω_{tors} of the profinite limit $\varprojlim_\alpha \Omega_\alpha$ is a finite group, and the quotient Ω of Ω by Ω_{tors} is topologically isomorphic to $\mathbf{Z}_p^{\delta_{\mathfrak{p}}}$ for $\delta_{\mathfrak{p}} = [F_{\mathfrak{p}} : \mathbf{Q}_p]$ the residue degree. Let us for each exponent $\beta \geq 0$ write Γ_β to denote the Galois group $\text{Gal}(K(\zeta_{p^{\beta+1}})/K)$, where $K(\zeta_{p^{\beta+1}})$ denotes the extension obtained from K by adjoining a primitive $p^{\beta+1}$ -th root of unity $\zeta_{p^{\beta+1}}$, and $\Gamma = \varprojlim_\beta \Gamma_\beta$ the profinite limit. The torsion subgroup Γ_{tors} is also finite, and the quotient Γ of Γ by Γ_{tors} topologically isomorphic to \mathbf{Z}_p . Writing

$$R_\infty^{(\mathfrak{p})} = \bigcup_{\alpha, \beta \geq 0} K[\mathfrak{p}^\alpha]K(\zeta_{p^\beta})$$

to denote the compositum of each of the $K[\mathfrak{p}^\alpha]$ and $K(\zeta_{p^\beta})$ of K , we shall take

$$\mathcal{G} = \text{Gal}(R_\infty^{(\mathfrak{p})}/K) \cong \Omega \times \Gamma = \varprojlim_{\alpha, \beta \geq 0} \Omega_\alpha \times \Gamma_\beta.$$

Note that for this choice of $\mathcal{G} = \text{Gal}(R_\infty^{(\mathfrak{p})}/K) \cong \Omega \times \Gamma$, we have an injection of completed group rings

$$(24) \quad \mathcal{O}[[\mathcal{G}]] \longrightarrow \bigoplus_{\mathcal{W}_0 \in \mathcal{G}_{\text{tors}}^\vee} \mathcal{O}[[G]], \quad \lambda \longmapsto (\mathcal{W}_0(\lambda))_{\mathcal{W}_0 \in \mathcal{G}_{\text{tors}}^\vee}.$$

Here, the sum runs over characters $\mathcal{W}_0 = \rho_0 \psi_0 = \rho_0(\chi_0 \circ \mathbf{N})$ of the torsion subgroup $\mathcal{G}_{\text{tors}} = \Omega_{\text{tors}} \times \Gamma_{\text{tors}}$ of \mathcal{G} , and each $\mathcal{W}_0(\lambda)$ denotes the specialization of the given element $\lambda \in \mathcal{O}[[\mathcal{G}]]$ to the character \mathcal{W}_0 of $\mathcal{G}_{\text{tors}}$, but *not* to any character of the profinite group $G = \Omega \times \Gamma \cong \mathbf{Z}_p^{\delta_{\mathfrak{p}}+1}$. Thus, the element $\mathcal{W}_0(\lambda)$ defines a genuine element of the completed group ring $\mathcal{O}[[G]]$ rather than just a value in \mathcal{O} . When $\mathcal{G}_{\text{tors}}$ of \mathcal{G} has order prime to p , this injection (24) is also a surjection.

Note as well that in this setting with $\mathcal{G} = \text{Gal}(R_\infty^{(\mathfrak{p})}/K) \cong \Omega \times \Gamma$, we can interpret each completed group ring $\mathcal{O}[[G]]$ as a multivariable power series ring in the following standard way. Let us for simplicity write $r \geq 2$ to denote the integer defined by $r = \delta_{\mathfrak{p}} + 1 = [F_{\mathfrak{p}} : \mathbf{Q}_p] + 1$. Fix a system of topological generators $\gamma_1, \dots, \gamma_r$ of the profinite group $G = \Omega \times \Gamma \cong \mathbf{Z}_p^r$. We then have the standard non canonical isomorphism to the formal power series ring $\mathcal{O}[[T_1, \dots, T_r]]$ in r indeterminates T_1, \dots, T_r ,

$$(25) \quad \mathcal{O}[[G]] \longrightarrow \mathcal{O}[[T_1, \dots, T_r]], \quad (\gamma_1, \dots, \gamma_r) \longmapsto (T_1 + 1, \dots, T_r + 1).$$

5.3.2. Multivariable p -adic L -functions. Let \mathcal{O} be a finite extension of \mathbf{Z}_p containing all of the eigenvalues of π . We have in this setting the following construction of p -adic L -functions $\mathcal{L}_{\mathfrak{p}} = \mathcal{L}_{\mathfrak{p}}(\pi, R_{\infty}^{(\mathfrak{p})})$ in $\mathcal{O}[[\mathcal{G}]]$, due to Hida [10], which extends constructions given for special cases by Panchishkin [21] and Dabrowski [6]. This p -adic L -function also generalizes those given by Hida [9] and Perrin-Riou [22] for $F = \mathbf{Q}$ (i.e. where $\delta_{\mathfrak{p}} = 1$ and π is a holomorphic discrete series of parallel weight $k \geq 2$). In all cases, the $r = (\delta_{\mathfrak{p}} + 1)$ -variable p -adic L -function $\mathcal{L}_{\mathfrak{p}} = \mathcal{L}_{\mathfrak{p}}(\pi, R_{\infty}^{(\mathfrak{p})}) \in \mathcal{O}[[\mathcal{G}]]$ is given by a profinite system of one-variable measures on the cyclotomic Galois groups $\Gamma^{(\alpha)} = \varprojlim_{\beta \geq 0} \Gamma_{\beta}^{(\alpha)}$ defined by

$$\mathrm{Gal}(K[\mathfrak{p}^{\alpha}](\zeta_{p^{\infty}})/K[\mathfrak{p}^{\alpha}]) = \varprojlim_{\beta \geq 0} \mathrm{Gal}(K[\mathfrak{p}^{\alpha}](\zeta_{p^{\beta}})/K[\mathfrak{p}^{\alpha}]).$$

Here, each $K[\mathfrak{p}^{\alpha}](\zeta_{p^{\beta}})$ denotes the extension obtained by adjoining to the fixed ring class extensions $K[\mathfrak{p}^{\alpha}]$ of K a primitive p^{β} -th power root of unity $\zeta_{p^{\beta}}$, i.e. so that $\Gamma_{\beta}^{(\alpha)} = \mathrm{Gal}(K[\mathfrak{p}^{\alpha}](\zeta_{p^{\beta}})/K[\mathfrak{p}^{\alpha}])$. That is, the p -adic L -function $\mathcal{L}_{\mathfrak{p}} = \mathcal{L}_{\mathfrak{p}}(\pi, R_{\infty}^{(\mathfrak{p})})$ is constructed as a profinite system $\{\mathcal{L}_{\mathfrak{p}}^{(\alpha)}\}_{\alpha \geq 0} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$ of measures $\mathfrak{L}_{\mathfrak{p}}^{(\alpha)}$ on the profinite groups $\Gamma^{(\alpha)} = \mathrm{Gal}(K[\mathfrak{p}^{\alpha}](\zeta_{p^{\infty}})/K[\mathfrak{p}^{\alpha}])$. See the discussion in Perrin-Riou [22, §5.1] for instance for the simplest case of $F = \mathbf{Q}$.

Recall that the algebraicity theorem of Shimura [28] shows that the values

$$(26) \quad \mathcal{L}(1/2, \pi \times \mathcal{W}) = L(1/2, \pi \times \mathcal{W}) \langle \pi, \pi \rangle^{-1}$$

are algebraic, and moreover that they lie in the finite extension $\mathbf{Q}(\pi, \mathcal{W})$ of \mathbf{Q} . Let us fix an embedding of $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$, so that we view the values (26) as elements of $\overline{\mathbf{Q}}_p$.

Theorem 5.4 ([10], [9], [21], [6], ...). *Assume conditions (i), (ii), and (iii) above. There exists a measure $\mathcal{L}_{\mathfrak{p}} = \mathcal{L}_{\mathfrak{p}}(\pi, R_{\infty}^{(\mathfrak{p})})$ in $\mathcal{O}[[\mathcal{G}]]$ such that for any finite-order character \mathcal{W} of \mathcal{G} (viewed as a Hecke character), we have the interpolation formula*

$$(27) \quad \mathcal{W}(\mathcal{L}_{\mathfrak{p}}) = \eta(\pi, \mathcal{W}) \cdot \mathcal{L}(1/2, \pi \times \overline{\mathcal{W}}) \in \overline{\mathbf{Q}}_p.$$

Here, $\eta(\pi, \mathcal{W})$ is some algebraic number which does not vanish if the conductor of \mathcal{W} is nontrivial (and which can be given precisely), viewed as an element of $\overline{\mathbf{Q}}_p$, and $\mathcal{W}(\mathcal{L}_{\mathfrak{p}}) = \int_{\mathcal{G}} \mathcal{W}(\sigma) d\mathcal{L}_{\mathfrak{p}}(\sigma)$ is the specialization of $\mathcal{L}_{\mathfrak{p}}$ at \mathcal{W} . Moreover, this p -adic L -function is constructed as a profinite system $\{\mathcal{L}_{\mathfrak{p}}^{(\alpha)}\}_{\alpha \geq 0}$ of one-variable p -adic L -functions $\mathcal{L}_{\mathfrak{p}}^{(\alpha)} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$ satisfying the analogous interpolation properties.

Proof. See the construction given in Hida [10], which is more general than what we require here (as it also interpolates over Hida families). See also the constructions of [21, §2, Main Theorem], [6], [9], and [22, §5.1] for special cases. \square

5.3.3. Weierstrass preparation. We shall make use of the following classical result on formal power series rings in one indeterminate. Let R be any discrete valuation ring with maximal ideal \mathfrak{m}_R , and fix a uniformizer ϖ_R of R . Consider the formal power series ring $R[[T]]$ in the indeterminate T . Recall that a polynomial $g(T)$ in $R[T]$ is said to be *distinguished (or Weierstrass)* if it takes the form

$$g(T) = T^n + b_{n-1}T^{n-1} + \dots + b_0$$

for some integer $n \geq 1$, with each coefficient b_i lying in the maximal ideal \mathfrak{m}_R .

Proposition 5.5 (Weierstrass preparation). *Let $h(T) = \sum_{j \geq 0} a_j T^j$ be an element of the formal power series ring $R[[T]]$. If $h(T)$ is not identically zero, then it can be expressed uniquely as the product*

$$h(T) = u(T)g(T)\varpi_R^{\mu_R}$$

of some unit $u(T)$ in $R[[T]]$ times some distinguished polynomial $g(T)$ in $R[T]$ times some integer power $\mu_R \geq 0$ of the fixed uniformizer ϖ_R of R .

Proof. The result is very standard, see e.g. [14, Ch. IV, Theorem 9.2]. \square

Given a nonzero element $h(T) = u(T)g(T)\varpi_R^{\mu_R}$ of a formal power series ring $R[[T]]$ as above, the degree of the distinguished polynomial $g(T)$ is known as the *Weierstrass degree* of $h(T)$, and the positive integer $\mu = \mu_R$ as the μ -invariant.

5.3.4. *Nontriviality in the cyclotomic variable(s).* Taking $R = \mathcal{O}$ to be the discrete valuation ring, the Weierstrass preparation theorem (Proposition 5.5) allows us to deduce from the nonvanishing condition of Theorem 4.7 that each of the one-variable p -adic L -functions $\mathcal{L}_{\mathfrak{p}}^{(\alpha)} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$ used to construct the $\mathcal{L}_{\mathfrak{p}} \in \mathcal{O}[[\mathcal{G}]]$ of Theorem 5.4 above is not identically zero. Hence, each of the $\mathcal{L}_{\mathfrak{p}}^{(\alpha)} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$ has some finite Weierstrass degree $n_{\alpha} \geq 0$ say, and thus the complex central values $L(1/2, \pi \times \rho(\chi \circ \mathbf{N}))$ with ρ of conductor $c(\rho) = \mathfrak{p}^{\alpha}$ (fixed) can vanish for at most n_{α} Dirichlet characters χ of p -power conductor. More precisely, we derive the following.

Theorem 5.6. *Assume that π is a holomorphic discrete series of weight $(k_j)_{j=1}^d$ with each $k_j \geq 2$, and for simplicity that π is \mathfrak{p} -ordinary. Let $\rho \in X(\mathfrak{p})$ be any ring class character of K of \mathfrak{p} -power conductor. Then, for all but finitely many Dirichlet characters χ of p -power conductor, the central value $L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ does not vanish.*

Proof. Suppose that $\rho \in X(\mathfrak{p})$ has conductor $c(\rho) = \mathfrak{p}^{\alpha}$ for $\alpha \geq 0$ some integer. Consider $\mathcal{L}_{\mathfrak{p}}^{(\alpha)} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$ the associated one-variable constituent of the r -variable p -adic L -function $\mathcal{L}_{\mathfrak{p}} \in \mathcal{O}[[\mathcal{G}]]$ of Theorem 5.4 above. Observe that by the nonvanishing property of Theorem 4.7 above paired with the interpolation property of Theorem 5.4 above, this element $\mathcal{L}_{\mathfrak{p}}^{(\alpha)} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$ cannot be identically zero. Consider the image $(\psi_0(\mathcal{L}_{\mathfrak{p}}^{(\alpha)}))_{\psi \in (\Gamma_{\text{tors}}^{(\alpha)})^{\vee}}$ of this nonzero element $\mathcal{L}_{\mathfrak{p}}^{(\alpha)} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$ under the injection defined analogously to (24) above,

$$\mathcal{O}[[\Gamma^{(\alpha)}]] \longrightarrow \bigoplus_{\psi \in (\Gamma_{\text{tors}}^{(\alpha)})^{\vee}} \mathcal{O}[[\Gamma]], \quad \lambda \longmapsto (\psi_0(\lambda))_{\psi \in (\Gamma_{\text{tors}}^{(\alpha)})^{\vee}}.$$

That is, consider the collection of one-variable p -adic L -functions $\psi_0(\mathcal{L}_{\mathfrak{p}}^{(\alpha)}) \in \mathcal{O}[[\Gamma]]$. Observe that since the order of each ψ_0 is prime to p , the specialization values interpolated by each of these p -adic L -functions are Galois conjugate, and hence each of the $\psi_0(\mathcal{L}_{\mathfrak{p}}^{(\alpha)}) \in \mathcal{O}[[\Gamma]]$ is not identically zero. Fixing a topological generator γ of $\Gamma \cong \mathbf{Z}_p$, consider the image $\mathcal{L}_{\mathfrak{p}}^{(\alpha)}(\psi_0; T)$ of each of the p -adic L -functions $\psi_0(\mathcal{L}_{\mathfrak{p}}^{(\alpha)})$ under the non-canonical isomorphism $\mathcal{O}[[\Gamma]] \cong \mathcal{O}[[T]]$ defined by $\gamma \mapsto T + 1$. We can now apply Weierstrass preparation (Proposition 5.5) to each power series $\mathcal{L}_{\mathfrak{p}}^{(\alpha)}(\psi_0; T)$ in $\mathcal{O}[[T]]$. In particular, each $\mathcal{L}_{\mathfrak{p}}^{(\alpha)}(\psi_0; T)$ has some finite Weierstrass degree $n_{\psi_0, \alpha}$ say, and the sum of these degrees $n_{\alpha} = \sum_{\psi_0} n_{\psi_0, \alpha}$ bounds the number of possible Dirichlet characters χ of p -power conductor for which the specialization

$\chi \circ \mathbf{N}(\mathcal{L}_{\mathfrak{p}}^{(\alpha)}) = \rho\chi \circ \mathbf{N}(\mathcal{L}_{\mathfrak{p}})$ can vanish. Equivalently by the interpolation formula (27), the sum n_{α} bounds the number of possible Dirichlet characters χ of p -power conductor for which the complex central value $L(1/2, \pi\rho\chi \circ \mathbf{N})$ can vanish. \square

Corollary 5.7. *For all but finitely many Dirichlet characters χ of p -power conductor, the central values $L(1/2, \pi, \chi \circ \mathbf{N})$ and $L(1/2, \pi, \eta\chi \circ \mathbf{N})$ do not vanish.*

Proof. Consider the result above for $\alpha = 0$, in particular for the trivial ring class character $\rho = \mathbf{1}$. The result then follows directly from Artin formalism for quadratic basechange, i.e. from the fact that we have for each Dirichlet character χ the identification of L -functions $L(s, \pi \times \mathbf{1}(\chi \circ \mathbf{N})) = L(s, \pi, \chi \circ \mathbf{N})L(s, \pi, \eta\chi \circ \mathbf{N})$. \square

5.3.5. *Nontriviality in the anticyclotomic variable(s).* We now explain how to put together known basechange theorems with the existence of suitable L -functions (i.e. those described above) to derive nonvanishing results for the central values $L(1/2, \pi \times \rho\psi)$ in certain anticyclotomic families. The arguments we present here are a variation of those presented in the prequel paper [31] to deduce nonvanishing results for cyclotomic families, and are formally similar in some places. They are however completely independent of the related averaging results established in [32], which employs spectral and analytic methods exclusively.

Let us keep all of the notations above used to define the profinite Galois group $G = \Omega \times \Gamma \cong \mathbf{Z}_p^r$, where $r = \delta_{\mathfrak{p}} + 1 = [F_{\mathfrak{p}} : \mathbf{Q}_p] + 1$. To be more precise, let us write

$$\Omega = \varprojlim_{\alpha} \Omega_{\alpha} = \varprojlim_{\alpha} \text{Gal}(K_{\mathfrak{p}^{\alpha}}/K) \cong \mathbf{Z}_p^{\delta_{\mathfrak{p}}}$$

to denote the anticyclotomic Galois group, i.e. where each $K_{\mathfrak{p}^{\alpha}} \subset K[\mathfrak{p}^{\alpha}]$ denotes the extension of degree $\mathbf{N}\mathfrak{p}^{\alpha} = p^{\alpha\delta_{\mathfrak{p}}}$ of K contained in the ring class field $K[\mathfrak{p}^{\alpha}]$ of conductor \mathfrak{p}^{α} (with the convention that $K_{\mathfrak{p}^0} = K$). Similarly, let us write

$$\Gamma = \varprojlim_{\beta} \Gamma_{\beta} = \varprojlim_{\beta} \text{Gal}(K_{\beta}/K) \cong \mathbf{Z}_p$$

to denote the cyclotomic Galois group, i.e. where each $K_{\beta} \subset K(\zeta_{\mathfrak{p}^{\beta+1}})$ denotes the extension of degree p^{β} of K contained in the cyclotomic field $K(\zeta_{\mathfrak{p}^{\beta+1}})$. Given an integer $\alpha \geq 0$, let us then define the basechange cyclotomic Galois group

$$\Gamma^{(\alpha)} = \varprojlim_{\beta} \Gamma_{\beta}^{(\alpha)} = \varprojlim_{\beta} \text{Gal}(K_{\mathfrak{p}^{\alpha}}K_{\beta}/K_{\mathfrak{p}^{\alpha}}) \cong \mathbf{Z}_p.$$

Let us now consider the existence of relevant basechange liftings associated to our fixed cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\text{GL}_2(\mathbf{A}_F)$. That is, let us write $\pi_K = \text{BC}_{K/F}(\pi)$ to denote the basechange of π to K , which exists by the work of [15] (cf. also [1] and [27, Theorem 15]). Thus, π_K is an automorphic representation of $\text{GL}_2(\mathbf{A}_K)$. On the other hand, using transitivity of basechange lifting according to [27, §16, “1. Base change lifting is transitive”, p. 345], we can consider for each integer $\alpha \geq 1$ the lifting $\pi_{K_{\mathfrak{p}^{\alpha}}} = \text{BC}_{K_{\mathfrak{p}^{\alpha}}/K}(\pi_K)$ of π_K to $K_{\mathfrak{p}^{\alpha}}/K$, i.e. which exists by [14], [1] and transitivity of basechange⁵ (cf. [27, Theorem]). Thus, $\pi_{K_{\mathfrak{p}^{\alpha}}}$ is an automorphic representation of $\text{GL}_2(\mathbf{A}_{K_{\mathfrak{p}^{\alpha}}})$ which is defined by the transitive operation on (existing) basechange liftings

$$(28) \quad \text{BC}_{K_{\mathfrak{p}^{\alpha}}/F}(\pi) = \text{BC}_{K_{\mathfrak{p}^{\alpha}}/K}(\pi_K) = \text{BC}_{K_{\mathfrak{p}^{\alpha}}/K}(\text{BC}_{K/F}(\pi)).$$

⁵At least if the residue degree $\delta = \delta_{\mathfrak{p}} = [F_{\mathfrak{p}} : \mathbf{Q}_p]$ equals one.

This gives us the following so-called Artin formalism for the associated L -functions,

$$(29) \quad L(s, \pi_{K_{\mathfrak{p}^\alpha}}) = \prod_{\rho \in \text{Gal}(K_{\mathfrak{p}^\alpha}/K)^\vee} L(s, \pi_K, \rho) = \prod_{\rho \in \text{Gal}(K_{\mathfrak{p}^\alpha}/K)^\vee} L(s, \pi \times \rho).$$

Here, $L(s, \pi_{K_{\mathfrak{p}^\alpha}})$ denotes the standard L -function of $\pi_{K_{\mathfrak{p}^\alpha}}$, the product runs over ring class characters ρ of the Galois group $\text{Gal}(K_{\mathfrak{p}^\alpha}/K)$ (viewed as Hecke characters of K after composing with the global reciprocity map of K), $L(s, \pi_K, \rho)$ denotes the standard L -function of the automorphic representation π twisted by the Hecke character ρ of K , and $L(s, \pi \times \rho) = L(s, \pi \times \pi(\rho))$ denotes the Rankin-Selberg L -function of the cuspidal automorphic representation π of $\text{GL}_2(\mathbf{A}_F)$ times the automorphic representation $\pi(\rho)$ of $\text{GL}_2(\mathbf{A}_F)$ associated to the Hecke character ρ of K . Moreover, given any cyclotomic character $\psi = \chi \circ \mathbf{N}$ of K as considered in our discussion above, we have the following twisted Artin formalism corresponding to [27, “2. Base change is compatible with twisting ...”, p. 346],

$$(30) \quad L(s, \pi_{K_{\mathfrak{p}^\alpha}}, \psi \circ \mathbf{N}_{K_{\mathfrak{p}^\alpha}/K}) = \prod_{\rho \in \text{Gal}(K_{\mathfrak{p}^\alpha}/K)^\vee} L(s, \pi_K, \rho\psi) = \prod_{\rho \in \text{Gal}(K_{\mathfrak{p}^\alpha}/K)^\vee} L(s, \pi \times \rho\psi).$$

Here, $L(s, \pi_{K_{\mathfrak{p}^\alpha}}, \psi \circ \mathbf{N}_{K_{\mathfrak{p}^\alpha}/K})$ denotes the L -function of $\pi_{K_{\mathfrak{p}^\alpha}}$ twisted by the Hecke character of $K_{\mathfrak{p}^\alpha}$ defined by $\psi \circ \mathbf{N}_{K_{\mathfrak{p}^\alpha}/K}$ of $K_{\mathfrak{p}^\alpha}$ (where $\mathbf{N}_{K_{\mathfrak{p}^\alpha}/K}$ denotes the norm homomorphism from $K_{\mathfrak{p}^\alpha}$ to K), $L(s, \pi_K, \rho\psi)$ denotes the L -function of π_K twisted by the Hecke character of K defined by $\rho\psi$, and $L(s, \pi \times \rho\psi) = L(s, \pi \times \pi(\rho\psi))$ denotes the $\text{GL}_2(\mathbf{A}_F) \times \text{GL}_2(\mathbf{A}_F)$ Rankin-Selberg L -function of π times $\pi(\rho\psi)$.

Our aim in this final subsection is to consider the implication of this twisted Artin formalism (30) with the nonvanishing result of Corollary 5.7 for our family of cyclotomic p -adic L -functions $\mathcal{L}_{\mathfrak{p}}^{(\alpha)}$ described above. To fix ideas, consider the image $(\mathcal{W}_0(\mathcal{L}_{\mathfrak{p}}))_{\mathcal{W}_0} \in \bigoplus_{\mathcal{W}_0} \mathcal{O}[[G]]$ of the element $\mathcal{L}_{\mathfrak{p}} = \mathcal{L}_{\mathfrak{p}}(\pi, K) \in \mathcal{O}[[G]]$ described in Theorem 5.4 above under the injection of completed group rings (24). We shall look at a fixed p -adic L -function $L_{\mathfrak{p}} = L_{\mathfrak{p}}(\pi, K) = \mathcal{W}_0(\mathcal{L}_{\mathfrak{p}}) = \mathcal{W}_0(\mathcal{L}_{\mathfrak{p}}(\pi, K)) \in \mathcal{O}[[G]]$ in the discussion that follows, keeping the choice of character $\mathcal{W}_0 \in \mathcal{G}_{\text{tors}}^\vee$ implicit.⁶

Recall that we fix a system of topological generators $\gamma_1, \gamma_2, \dots, \gamma_{\delta+1}$ of G , where again $\delta = \delta_{\mathfrak{p}} = [F_{\mathfrak{p}} : \mathbf{Q}_p]$ denotes the residue degree. Let us take $\gamma_0 = \gamma_{\delta+1} \in \Gamma$ to be the fixed generator of the cyclotomic part, with $\gamma_1, \dots, \gamma_{\delta} \in \Omega$ the fixed system of generators of the anticyclotomic part. Let us also for each integer $\alpha \geq 1$ fix a topological generator $\gamma_0^{(\alpha)} \in \Gamma^{(\alpha)}$ in such a way that

$$\text{Res}_{K_{\mathfrak{p}^\alpha}/K} \left(\gamma_0^{(\alpha)} \right) = \gamma_0,$$

where $\text{Res}_{K_{\mathfrak{p}^\alpha}/K}$ denotes the (bijective) restriction homomorphism from $\Gamma^{(\alpha)}$ to Γ . Equivalently, composing with the appropriate inverse reciprocity maps of class field theory, we choose a system liftings of topological generators which are compatible under composition with the norm homomorphisms $\mathbf{N}_{K_{\mathfrak{p}^\alpha}/K}$ (cf. [31, (20)]). We shall consider the images of our fixed $L_{\mathfrak{p}} = L_{\mathfrak{p}}(\pi, K) \in \mathcal{O}[[G]]$ in the commutative

⁶In particular, the characters ρ of the profinite group Ω which we consider in our arguments will all be wildly ramified.

diagram

$$(31) \quad \begin{array}{ccc} \mathcal{O}[[\Gamma^{(\alpha)}]] & \xleftarrow{\text{Res}_{K_{\mathfrak{p}^\alpha}/K}^{-1}} & \mathcal{O}[[\Gamma^{(0)}]] \\ \downarrow & & \downarrow \\ \mathcal{O}[[\Gamma^{(\alpha)}]] & \xrightarrow{\text{Res}_{K_{\mathfrak{p}^\alpha}/K}} & \mathcal{O}[[\Gamma^{(0)}]] \\ \downarrow & & \downarrow \\ \mathcal{O}[[T]] & \xlongequal{\quad} & \mathcal{O}[[T]]. \end{array}$$

Here, the top horizontal arrows are the isomorphism determined by the restriction homomorphism and its inverse (cf. [31, (24)]). The bottom pair of vertical arrows are the non canonical bijections obtained by choosing topological generators. We argue as in [31, Lemma 2.5] that image of a nonzero element of $\mathcal{O}[[\Gamma^{(\alpha)}]]$ in this diagram has invariant Weierstrass degree. Thus, our task is to use the basechange relations (30) to relate the family of one-variable p -adic L -functions $L_{\mathfrak{p}}^{(\alpha)} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$, these elements being determined uniquely by their p -adic interpolation properties.

To simplify notation, let us for an integer $\alpha \geq 1$ and a cyclotomic Hecke character $\psi = \chi \circ \mathbf{N}$ of K write $\psi^{(\alpha)}$ to denote the cyclotomic Hecke character of $K_{\mathfrak{p}^\alpha}$ obtained via composition with the norm homomorphism $\mathbf{N}_{K_{\mathfrak{p}^\alpha}/K}$. Hence,

$$\psi^{(\alpha)} = \psi \circ \mathbf{N}_{K_{\mathfrak{p}^\alpha}/K} = \chi \circ \mathbf{N}_{K_{\mathfrak{p}^\alpha}/K},$$

where $\mathbf{N}_{K_{\mathfrak{p}^\alpha}/K}$ denotes the norm homomorphism from $K_{\mathfrak{p}^\alpha}$ to K .

Proposition 5.8. *Let $\alpha \geq 1$ be an integer, and $\psi^{(\alpha)}$ a basechange cyclotomic character of the form described above. The rule that sends such a character $\psi^{(\alpha)}$ to the value determined by $\psi^{(\alpha)}(L_{\mathfrak{p}}) = \psi \circ \mathbf{N}_{K_{\mathfrak{p}^\alpha}/K}(L_{\mathfrak{p}})$ defines an element of the completed group ring $\mathcal{O}[[\Gamma^{(\alpha)}]]$, and satisfies the following interpolation property (which in fact determines it uniquely):*

$$\begin{aligned} \psi^{(\alpha)}(L_{\mathfrak{p}}^{(\alpha)}) &:= \int_{\Gamma^{(\alpha)}} \psi^{(\alpha)}(\sigma) dL_{\mathfrak{p}}^{(\alpha)}(\sigma) \\ &= \prod_{\rho \in \text{Gal}(K_{\mathfrak{p}^\alpha}/K)^\vee} \rho(\psi(L_{\mathfrak{p}})) \\ &= \prod_{\rho \in \text{Gal}(K_{\mathfrak{p}^\alpha}/K)^\vee} \eta(\pi, \rho\psi) \cdot \mathcal{L}(1/2, \pi \cdot \rho\psi). \end{aligned}$$

Here, we have used the same notations and conventions as for Theorem 5.4 above. In particular, this element is the image of the $r = (\delta + 1)$ -variable p -adic L -function $L_{\mathfrak{p}} = L_{\mathfrak{p}}(\pi, K) \in \mathcal{O}[[G]]$ in $\mathcal{O}[[\Gamma^{(\alpha)}]]$. Moreover, as seen by taking the image of this element $L_{\mathfrak{p}}^{(\alpha)} \in \mathcal{O}[[\Gamma^{(\alpha)}]]$ in the commutative diagram (31), the Weierstrass degree of this element is independent of the choice of exponent α .

Proof. See [31, Lemma 2.6, Corollary 2.7], the same formal argument works here. The idea is to argue that composition with the restriction/norm homomorphism determines an \mathcal{O} -valued measure $dL_{\mathfrak{p}}^{(\alpha)}$ on the profinite Galois group $\Gamma^{(\alpha)}$ satisfying the stated interpolation property. Using the twisted Artin formalism (30) then allows us to view this as the image of $L_{\mathfrak{p}}$ in $\mathcal{O}[[\Gamma^{(\alpha)}]]$. \square

Thus, since the Weierstrass degree stabilises in the suitable sense by this key lemma, we obtain as a consequence the following result.

Theorem 5.9. *Fix a p -adic L -function $L_{\mathfrak{p}} = L_{\mathfrak{p}}(\pi, K) = \mathcal{W}_0(\mathcal{L}_{\mathfrak{p}}(\pi, K)) \in \mathcal{O}[[G]]$, following the discussion given above. There exists in this setting an effectively computable integer $\beta_0 \geq 0$ such that for all cyclotomic characters $\psi = \chi \circ \mathbf{N} \in X(\mathfrak{p})$ of conductor greater than \mathfrak{p}^{β_0} , the central value $L(1/2, \pi \times \rho \mathcal{W}_0 \psi)$ does not vanish for any finite-order character ρ factoring through the anticyclotomic Galois group $\Omega \cong \mathbf{Z}_p^{\delta}$.*

Proof. The result follows directly from Proposition 5.8 after using Weierstrass preparation with the twisted Artin formalism (30). \square

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MAX PLANCK INSTITUTE FOR MATHEMATICS, VIVATSGASSE 7, BONN 53111, GERMANY
E-mail address: `vanorder@mpim-bonn.mpg.de`